VOLUME XLVIII

The Mathematics Teacher

NUMBER FIVE

MAY 1955

The official journal of
THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The Mathematics Teacher *is a journal of The National Council of Teachers of Mathematics devoted to the interests of mathematics teachers in the Junior High Schools, Senior High Schools, Junior Colleges and Teacher Education Colleges.*

All editorial correspondence, including books for review, should be addressed to the Editor.

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THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1201 Sixteenth Street N. W., Washington 6, D. C.

Printed at Menasha, Wisconsin, U.S.A. Entered as second-class matter at the post office at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 412 P. L. & R., authorized March 1, 1930. Printed in U.S.A.

The Mathematics Teacher

volume XLVIII, number 5 May 1955

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Some concepts of logic and their application in elementary mathematics

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Many mathematicians think that a more modern mathematics must be introduced into the high schools and colleges. This means that some consideration must be given to basic principles.

BEFORE ANY HEADWAY can be made in a study of a mathematical logic certain fundamental words and symbols must be carefully defined. Sometimes these definitions are quite different from what psychologically you would like to believe; that makes initial work in logic difficult but not impossible.

One word that is used in a special way is the word *statement*. A statement in its simplest form is a simple declarative sentence that is either true or false, but not both. In general the statements that are made in a symbolic logic deal exclusively with mathematical or logical ideas. "All cats catch mice" is a statement; it will be convenient to refer to such a statement by a single letter "P" in the symbolic logic. Then "P" is called the *translation* of the statement "All cats catch mice." Notice that you can interpret the statement to mean an assertion that a class or an individual is included in another class. For example, we can *convert* "All cats catch mice" into the statement "All cats are animals that catch mice." The converted statement asserts that the class of cats is included in the class of animals that catch mice.

SOME SYMBOLS OF LOGIC

New statements are constructed by means of *statement connectives*. The statement connectives employed to form statement functions and the usual ways of reading them are given in the following table.

Symbol	*Name*	*Read as*
$\sim$	curl or tilde	not
&	ampersand	and
$\vee$	vel	or
$\rightarrow$		implies

If "P" is a translation of "12 is exactly divisible by 3" and "Q" is a translation of "12 is exactly divisible by 4," then "P & Q" is a translation of "12 is exactly divisible by 3 *and* 12 is exactly divisible by 4." Some writers would say that unfortunately the latter statement is often converted into "12 is exactly divisible by 3 and 4" with a resulting loss in clarity of meaning. For the most part, however, this sort of conversion of statements in mathematics does not trouble us. The meaning is clear.

The particular statement function that is needed for understanding what follows is "$P \rightarrow Q$". A statement of a conditional may have several forms. Two common forms appear in the following statements from plane geometry: "If x and y are angles having their sides parallel, then $x = y$" or "x and y being angles and having their sides parallel implies $x = y$." Such statements are translated as "$P \rightarrow Q$" in the symbolic logic, where "P" is a translation

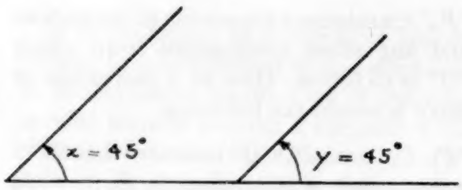

Figure 1

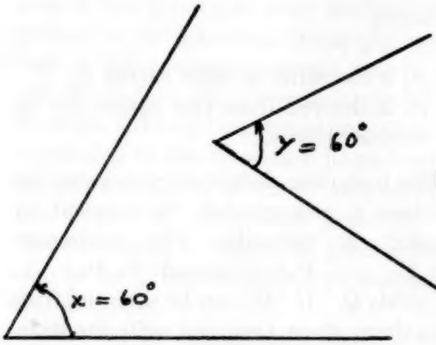

Figure 2

of the statement "the angles x and y have their sides parallel" and "Q" is a translation of the statement "$x = y$."

By definition "$P \rightarrow Q$" is a statement if "P" and "Q" are statements; "$P \rightarrow Q$" is a true statement except when "P" is true and "Q" is false.

Let us examine the definition of the conditional and apply it to the foregoing statement from geometry "If the angles x and y have their sides parallel, then $x = y$." Before considering meanings for this statement, we remark that the statement concerns a particular pair of angles x and y; it does not refer to all pairs of angles x and y. Now, if the given pair of angles are the angles pictured in Figure 1, then "$P \rightarrow Q$" is a true statement because both "P" and "Q" are true statements. For Figure 2, "$P \rightarrow Q$" is a true statement in spite of the fact that "P" is false. In the case of Figure 3, "$P \rightarrow Q$" is a false statement according to the definition because "P" is true and "Q" is false. The last case is shown in Figure 4; "$P \rightarrow Q$" is a true statement, even though both "P" and "Q" are false.

Some of the true cases of "$P \rightarrow Q$" may seem strange to you, but these cases are required so that however truth values are assigned to "P" and "Q," the truth value of "$P \rightarrow Q$" is determined. These cases may be considered as additional to the familiar cases, since they are surely not in conflict with them. Since "P" and "Q" each may assume the truth values T (true) and F (false), there are four ways in which a truth value of "P" can be combined with a truth value of "Q" in "$P \rightarrow Q$." A truth table for the conditional follows.

P	Q	$P \rightarrow Q$
T	T	T
F	T	T
T	F	F
F	F	T

In order to make full use of the symbolic logic, there are needed some rules by which new results can be inferred or deduced. The only such rule that is consid-

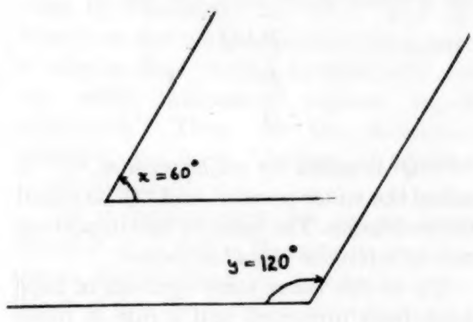

Figure 3

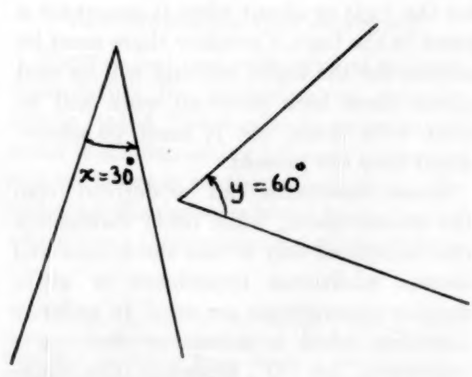

Figure 4

ered here is, perhaps, the most fundamental. As an illustration of the use of this rule of inference, consider the theorem "If a triangle has two sides equal, then the angles opposite these sides are equal" as having been previously proved. Written in symbolic form, the theorem states

$$((ABC \text{ is a triangle}) \ \& \ (AC=BC)) \rightarrow (\angle A = \angle B).$$

Now, suppose you wish to use this theorem to deduce a result; in some way you prove that

$$(ABC \text{ is a triangle}) \ \& \ (AC=BC);$$

then you *infer* from the theorem and the foregoing information the statement "$\angle A = \angle B$". That is, knowing the theorem is a proved statement and having proved in some way that two sides of a triangle are equal, you infer or deduce that a particular pair of angles of the triangle in question are equal. Written compactly in the symbols of the logic, the rule is

$$\begin{array}{l} P \rightarrow Q \\ \underline{P \quad\quad} \\ \therefore Q \end{array}$$

"$P \rightarrow Q$" is called the *major premise*, "P" is called the *minor premise*, and "Q" is called the *conclusion*. The label for this important rule of inference is *modus ponens*.

Up to this point some symbols of logic have been presented and a rule of inference. Nothing has been said about axioms for the logic or about what is meant by a proof in the logic. Certainly there must be axioms for the logic; nothing will be said about them here since no work will be done with them, but it must be understood they are present.

Some statements can be derived from the axioms alone, while other statements can be derived only in case the axioms and certain additional hypotheses or given data or assumptions are used. In order to formalize what is meant by deriving a statement, let "Q" represent the statement to be proved and "P_1," "P_2," $\cdots$, "P_n" translations for axioms of the system and any other assumptions from which "Q" is to follow. Then by a derivation or proof is meant the following.

"$P_1, P_2, \cdots, P_n \vdash Q$" indicates that there is a sequence of statements $S_1, S_2, \cdots, S_s$ such that S_s is Q and for each S_i either:

(1) S_i is [in the form of] an axiom [of the logic].
(2) S_i is a P.
(3) S_i is the same as some earlier S_j.
(4) S_i is derived from two earlier S's by modus ponens."[1]

The foregoing definition of a proof introduces a new symbol "$\vdash$" called by Rosser[2] a turnstile. The statement "$P_1, P_2, \cdots, P_n \vdash Q$" is read "$P_1, P_2, \cdots, P_n$ yields Q." If "Q" can be deduced from the axioms alone, then you write the statement as "$\vdash Q$" and read it "yields Q."

The Deduction Principle

The point has been reached where it is possible to state the deduction principle, one of the two concepts of logic that will be discussed in this paper. Enough symbols have been presented to make the symbolic statement understandable.

Deduction principle. If $P_1, P_2, \cdots, P_n, Q \vdash R$, then

$$P_1, P_2, \cdots, P_m \vdash Q \rightarrow R.$$

The significance of this principle is clearer when it is stated in a less general form: If $Q \vdash R$, then $\vdash Q \rightarrow R$. That is, if on the basis of the axioms and the assumption of "Q" there exists a correct deduction of "R," then there exists a correct deduction of "Q implies R" based on the axioms alone.

[1] J. Barkley Rosser, *Logic for Mathematicians* (New York: McGraw-Hill Book Co., 1953), p. 56.

[2] The symbol "$\vdash$" was first introduced by Gottlob Frege, *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*. Halle: Nebert, 1879. Its use as indicated above is due to J. Barkley Rosser, "A Mathematical logic without Variables," *Duke Mathematical Journal*, vol. I (1935), pp. 328–355, and modified by Stephen C. Kleene, "Proof by Cases in Formal Logic," *Annals of Mathematics*, 2nd series, vol. XXXV (1934), pp. 529–544.

It is important to understand the deduction principle because it is so often used in mathematical proofs. One of the commonest ways of proving a theorem of the "if—then—" type is to assume the "if" part, or hypothesis as true, then to reason correctly to the "then" part, or conclusion. Of course, it is usually necessary also to use axioms of some mathematical system in the reasoning. A careful inspection of this procedure reveals that it does not prove that the theorem itself follows from the axioms, but proves only that the conclusion of the theorem follows from its hypothesis and the axioms. To assume that this procedure constitutes proof of the theorem itself is tacitly to assume some step in the proof equivalent to the deduction principle.

Consider, for example, the theorem from plane geometry: If a figure is a quadrilateral, then whenever its opposite sides are equal, the figure is a parallelogram. The usual proof begins by drawing (Fig. 5)

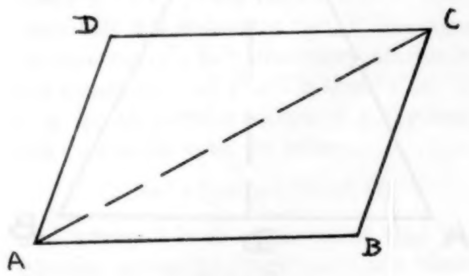

Figure 5

a diagonal of the quadrilateral $ABCD$, states that the two pairs of opposite sides AB and DC, AD and BC are equal, then goes on to show that triangles ABC and ADC are congruent, and from this gets opposite sides of the quadrilateral parallel; thus, it is deduced the quadrilateral is a parallelogram. Notice that this proof parallels the statement of the deduction principle. We might take "P" as a translation of "$ABCD$ is a quadrilateral," "Q" as a translation of "$AB=DC$ and $AD=BC$," "R" as a translation of "$ABCD$ is a parallelogram." Then we have "Axioms of geometry, P, $Q \vdash R$" as a translation of "The axioms of geometry, $ABCD$ is a quadrilateral, $AB=DC$ and $AD=BC$, yields $ABCD$ is a parallelogram." By using the deduction principle once, we have "Axioms of geometry, $P \vdash Q \rightarrow R$"; a second use of the deduction principle brings us to "Axioms of geometry $\vdash P \rightarrow (Q \rightarrow R)$." The last statement indicates that there exists a demonstration of the theorem on the basis of the axioms of geometry. "$P \rightarrow (Q \rightarrow R)$" is a symbolic statement, of course, of the given theorem.

The same deduction principle is used in algebra. An example of the use of the deduction principle occurs in the process of finding the solution of an equation in one unknown. If the equation in question is "$2x-3=5$," then we first prove that "$2x-3=5 \rightarrow x=4$," and then verify that 4 is indeed a value that satisfies the equation; that is, "$2x-3=5$ when $x=4$." In carrying out the first part, we habitually start by assuming "$2x-3=5$" and deduce from this with the help of the axioms of algebra that "$x=4$." Symbolically, we can write "Axioms of algebra, $2x-3=5 \vdash x=4$." Then, by the deduction principle, we have "Axioms of algebra $\vdash (2x-3=5) \rightarrow (x=4)$."

The deduction principle as we have stated it is in fact a theorem in an axiomatic development of symbolic logic. Proofs of this theorem can be found in several places in the literature.[3]

The generalization principle

There is just one more symbol to introduce before the generalization principle can be stated. The symbol in question will assist in distinguishing between unknowns and variables in mathematical usage. The following discussion is based upon Rosser[4] who gives a clear exposition of the distinction.

[3] Rosser, *op. cit.*, p. 75; Alonzo Church, *Introduction to Mathematical Logic* (Princeton, N. J.: Princeton University Press, 1944), p. 10; Stephen C. Kleene, *Introduction to Mathematics* (New York: D. Van Nostrand, 1952), p. 90.

[4] Rosser, *op. cit.*, pp. 82–94.

Mathematicians use the letters of the alphabet to denote both unknowns and variables. No difference in the letters appears to the eye. A reader must judge for himself if a writer intends to use a letter as an unknown or as a variable. Clearly, when you write $x^2-5x+4=0$, you are using x to represent an unknown quantity (number) whose value is to be determined. If you write

$$x^2-1=(x+1)(x-1),$$

you probably intend x to be a variable. In simple cases like these two examples it is easy to decide which usage a writer intends his reader to understand. In the first example the statement is true for two values of the unknown, 4 and 1. In the second example the statement is true for all values of x, that is, for all real (or complex) number values of the variable.

A way to distinguish the two uses of a letter that is used in algebra is to use the symbol "$\equiv$" when the letter represents a variable. For example, $x^2-5x+4=0$ indicates that x represents an unknown but fixed number whose value must be determined. If we write $x^2-1\equiv(x+1)(x-1)$, we intend to give a reader the information that x is a variable whose values range over all complex numbers. But this device is limited to statements of equality. A symbolic logic, depending upon form alone, needs some way for indicating whenever a letter is used whether it is an unknown or a variable, whether it is used in a statement of equality or not.

An unknown, unlike a variable, is not supposed to vary. If we start out to solve $x^2-5x+4=0$, we go from step to step as follows,

$$x^2-5x+4=0$$

$$(x-4)(x-1)=0$$

$$x-4=0 \quad \text{or} \quad x-1=0$$

$$x=4 \quad \text{or} \quad x=0,$$

and x is supposed to be the same in each step. Just consider how upsetting it would be if our x shifted value from one step to another!

Actually the same procedure is used in proving an identity like $\tan^2 x+1=\sec^2 x$. We choose some unknown or representative, but fixed, value of x and show that the equality is satisfied. Our next step is to state, either to ourselves or in words, "since the equality is satisfied for a representative value of x, then it must be satisfied for all values of x." It seems as if we work first with x as an unknown, prove the statement $\tan^2 x+1=\sec^2 x$, and then replace x as an unknown by x as a variable.

Consider an example from geometry. Suppose we wish to prove the theorem: If two sides of a triangle are equal, the angles opposite these sides are equal. The proof proceeds somewhat as follows.

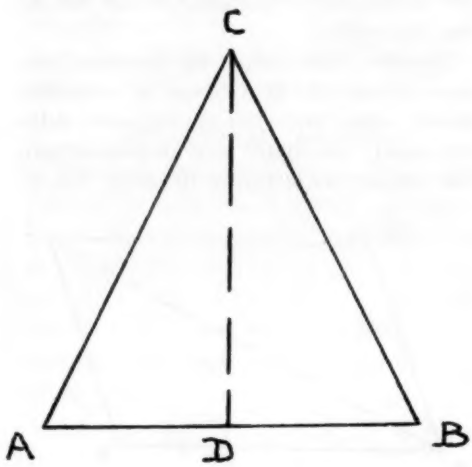

Figure 6

Given data: Triangle ABC with $AC=BC$ (Fig. 6).
Problem: To prove $\angle A=\angle B$.
Construction: Bisect AB and draw line DC.
Proof: In triangles ADC and BDC, $AC=BC$ (given); $CD=CD$ (identity); $AD=DB$ (construction). Hence, triangle ADC is congruent to triangle BDC and $\angle A=\angle B$.

To quote Rosser, "Clearly, it would ruin this proof completely if halfway through one should permit the triangle ABC to change into some other isosceles triangle, say one in which sides AB and BC were

equal instead of sides AC and BC. So the proof is carried out for some fixed (but undetermined) isosceles triangle, and only after the proof is complete, and the theorem proved, do we permit the replacement of our fixed, unknown triangle by a variable triangle."[5]

As in the case of the proof of the trigonometric identity, the proof is made as though you were working with an unknown. After the proof is made, the unknown is replaced by a variable. ONLY IN A PROVED STATEMENT CAN AN UNKNOWN BE REPLACED BY A VARIABLE.

In our symbolic logic we want to use unknowns and variables as mathematicians use them, but we must have a way of distinguishing the usage on the basis of form alone; meanings are taboo in our symbolic logic. For this purpose (and our purposes in this section) the symbol "(x)" is introduced. The symbol "(x)" is interpreted to denote any one of: "For all $x, \cdots$"; "For every $x, \cdots$"; "For each $x, \cdots$." Consequently if a statement "$P(x)$" has some occurrences of x in it, the interpretation of the statement "$(x)\ P(x)$" is that "$P(x)$" is true for all possible values of x. To consider a specific case, we write

$$(x)((x^2-1)=(x+1)(x-1))$$

and interpret it as a statement that the equality is true for every value of x whatsoever.

Generalization principle. If $P_1, P_2, \cdots, P_n, Q$ are statements, not necessarily distinct, and x is a variable that does not appear as an unknown in any of $P_1, P_2 \cdots, P_n$, and if $P_1, P_2, \cdots, P_n \vdash Q$, then $P_1, P_2, \cdots, P_n \vdash (x)Q$.

Or, in less general form, the generalization principle is:

If $P(x)$ is a statement and x is a variable and $\vdash P(x)$, then $\vdash (x)\ P(x)$.

The generalization principle states that if "$P(x)$" is proved, then you can infer "$(x)P(x)$," that is, "for all x, $P(x)$." In other words you can substitute a variable for an unknown in a *proved statement.*

Now, let us show some examples of the use of the generalization principle in algebra and geometry. First, let us return to the theorem on isosceles triangles. If "P" is taken as a translation of "If a triangle has two sides equal, the angles opposite these sides are equal," the proof given is of the form "$\vdash P(x)$," where the variable involved refers to the isosceles triangle. Hence, a statement concerning an isosceles triangle is deduced; from this statement we obtain by generalization on the triangle "$\vdash (x)P(x)$," a statement about all isosceles triangles.

The generalization principles requires that it is possible to deduce correctly a statement "$P(x)$"; from this fact, the principle says, it is possible to generalize on x to "$(x)P(x)$."

The derivation of the formula for the roots of a quadratic equation is an example from algebra of direct application of the generalization principle. When you present this proof to your students, you begin with $ax^2+bx+c=0$, $a\neq 0$, and fixed but unknown complex number values for a, b, c, and x. You proceed step by step,

$$4a^2x^2+4abx+b^2=b^2-4ac,$$

$$(2ax+b)^2=b^2-4ac$$

$$2ax+b=\pm\sqrt{b^2-4ac}$$

to the conclusion

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

and leave your students with the impression that they now have a formula that is true for all a, b, c, and x when these letters represent any complex numbers. Clearly, you have correctly deduced the formula from the axioms of algebra and the assumption of the quadratic equation. By the generalization principle, then, the formula is true for all complex number values of a, b, c, and x. Symbolically you have,

[5] Rosser, *op. cit.*, pp. 86–87.

Axioms of algebra,

$$a \neq 0, ax^2+bx+c=0 \vdash x = \frac{-b \pm \sqrt{b^2-4ac}}{2a};$$

by the deduction principle,

Axioms of algebra

$$\vdash ((a \neq 0) \ \& \ (ax^2+bx+c=0)) \rightarrow \left(x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \right)$$

Finally, the generalization principle allows you to write,

Axioms of algebra

$$\vdash (a, b, c, x)((a \neq 0) \ \& \ (ax^2+bx+c=0) \rightarrow \left(x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \right)$$

Rosser[6] mentions the theorem "If three parallel lines cut off equal segments on one transversal, then they cut off equal segments on any transversal" as an example of a statement that looks like an instance of the generalization principle but is not. The question is, why not? Are you not asked to prove a statement about a representative transversal and from this to conclude that the statement is true for all transversals? Let us examine more carefully what is involved here. Superficially it looks as if, first, we have the parallel lines cutting off equal segments on an arbitrary (unknown) transversal; second, we generalize, replacing the unknown transversal by a variable transversal. However, if we really proceeded in this fashion, we should be led to the statement: If three lines are parallel, then they cut off equal segments on every transversal. Clearly, this statement is absurd.

On the other hand we know that the foregoing theorem is a proved generality about properties of parallels and transversals. It must be possible to analyze the theorem in such a way that the generalization principle can be used properly. Let us deal with the problem by means of the following symbols:

$P(a, b, c)$ is a translation of "Lines a, b, c are mutually parallel";

$T(y, a, b, c)$ is a translation of "y is a transversal of a, b, c";

$S(z, a, b, c)$ is a translation of "a, b, c, cut off equal segments on z."

Referring to Figure 7, let "Q" be a translation of "a, b, c are mutually parallel, m is a transversal of a, b, c, and a, b, c cut off equal segments on m." In terms of our notation, we have

$$Q: P(a, b, c) \ \& \ T(m, a, b, c) \ \& \ S(m, a, b, c).$$

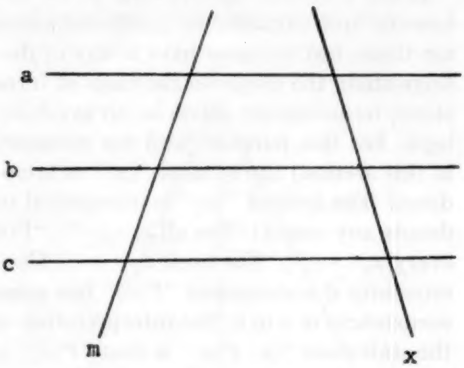

Figure 7

Now "Q" is not a proved statement; we may not generalize on "m" or on any other unknown in it.

However, in proving the theorem, we normally start by demonstrating,

$$Q, T(x, a, b, c) \vdash S(x, a, b, c).$$

We may not use the generalization principle here on "x" because of the presence of "x" in "$T(x, a, b, c)$," which appears to the left of the turnstile. Using the deduction principle, we can obtain.

$$Q \vdash T(x, a, b, c) \rightarrow S(x, a, b, c).$$

Now, since "x" does not occur as an unknown in any statement to the left of the turnstile, it is possible to generalize on "x" to get,

$$Q \vdash (x)(T(x, a, b, c) \rightarrow S(x, a, b, c)).$$

[6] Rosser, *op. cit.*, p. 104.

By a second application of the deduction principle, we get,

$$\vdash Q \rightarrow (x)(T(x, a, b, c) \rightarrow S(x, a, b, c)).$$

A more or less direct translation of the foregoing symbolic statement is:

If a, b, c are mutually parallel lines and cut off equal segments on transversal m, then for all lines x, whenever x is a transversal, a, b, c, cut off equal segments on it.

In generalizing "W" to "$(x)W$," the restriction that "x" not appear as an unknown in any of the assumptions used to prove W is important enough to warrant a further example. Consider the theorem from solid geometry,

> All lines perpendicular to a given line b at a point A lie in the plane π which is perpendicular to b at A.

Normally, we start a demonstration of this theorem by assuming,

$$H: \pi \perp b \text{ at } A,\ x \perp b \text{ at } A \quad \text{(where } x \text{ is a line)},$$

and demonstrate,

$$H \vdash x \text{ lies in } \pi.$$

If we use the generalization principle at this point, we obtain a statement that can be translated as:

$$\pi \perp b \text{ at } A \quad \text{and}$$
$$x \perp b \text{ at } A \text{ yields all lines lie in } \pi,$$

which is absurd. Of course, the trouble is that "H" depends on "x."

What we must do is use the deduction principle, first, in order to obtain a statement in which the unknown "x" does not appear to the left of the turnstile. Then, and only then, will it be possible to use the generalization principle. Hence, we have, after one use of the deduction principle,

$$\pi \perp b \text{ at } A \vdash x \perp b \text{ at } A \rightarrow x \text{ lies in } \pi;$$

now it is all right to generalize on "x"; we have,

$$\pi \perp b \text{ at } A \vdash (x)(x \perp b \text{ at } A \rightarrow x \text{ lies in } \pi),$$

and by deduction a second time,

$$\vdash \pi \perp b \text{ at } A \rightarrow (x)(x \perp b \text{ at } A \rightarrow x \text{ lies in } \pi).$$

A translation of the foregoing symbolic statement is,

If plane π is perpendicular to line b at A, then for all lines x, whenever x is perpendicular to b at A, x lies in π,

which is a paraphrase of the given theorem.

Conclusion

In our teaching of elementary mathematics much emphasis today is placed upon teaching for meaning and understanding. One might ask: If meanings are important for students, are they not even more important for the teachers of these students? Before a teacher can give correct meanings of mathematical operations and concepts to his students, he must have correct meanings of these mathematical operations and concepts himself. Since so much emphasis is placed upon deductive proof in plane geometry it seems that a teacher ought to know and to understand the logic of deductive proof.

Teachers are horrified at teaching that is based on imitation as a theory of learning. But just how does one in his mathematical training learn how to make a correct proof? Unless the teaching of plane geometry in high school classrooms is entirely different from plane geometry textbooks, students are learning how to construct a correct proof by examining model proofs, in short, by imitation.

One of the emerging trends in secondary school mathematics and elementary college courses is a shift away from purely manipulative mathematics. More stress than formerly is being given to mathematical logic and the foundations of mathematics. One reason for this new emphasis in subject matter is a belief that students get a better idea of what mathematics is from a study of its more modern aspects than they would from a study of just college algebra, for example. Another reason for the shift is that applications of

mathematics stress use of fundamental abstract concepts more often than straightforward manipulation.

What can we as teachers do about this? We can learn more mathematics. In fact, it will be necessary for us to learn new mathematics in order to keep up with textbooks and new applications. Should teachers introduce the symbols of mathematical logic and some of its principles in secondary school mathematics? The answer to the question depends upon the circumstances. We believe that teachers and textbooks should do a better job of introducing high school students to the concepts of logic. This might take the form of using language correctly and carefully or by making a restatement of a theorem, or a step in a proof, to bring out in words use of the deduction principle or the generalization principle. There is no intent here to propose that a teacher develop the latter principles as in this paper or name them for high school students, but it is suggested that teachers know these principles and capitalize on what values there are in learning by imitation. If a teacher uses language and ideas correctly, his students are impelled along the road to using the same language and ideas correctly.

"For the human mind the absolute continuity of motion is inconceivable. The laws of motion of any kind only become comprehensible to man when he examines units of this motion, arbitrary division of continuous motion into discontinuous units that a great number of human errors proceeds.

"We all know the so-called sophism of the ancients, proving that Achilles would never overtake the tortoise, though Achilles walked ten times as fast as the tortoise. As soon as Achilles passes over the space separating him from the tortoise, the tortoise advances one-tenth of that space. Achilles passes over that tenth, but the tortoise has advanced a hundredth, and so on to infinity. This problem seemed to the ancients insoluble. The irrationality of the conclusion (that Achilles will never overtake the tortoise) arises from the arbitrary assumption of disconnected units of motion, when the motion both of Achilles and the tortoise was continuous.

"By taking smaller and smaller units of motion we merely approach the solution of the problem, but we never attain it. It is only by assuming an infinitely small magnitude, and a progression rising from it up to a tenth, and taking the sum of that geometrical progression, that we can arrive at the solution of the problem. A new branch of mathematics, dealing with infinitely small quantities, gives now in other more complex problems of dynamics solutions of problems that seemed insoluble.

This new branch of mathematics, unknown to the ancients, by assuming infinitely small quantities, that is, such as secure the chief condition of motion (absolute continuity), corrects the inevitable error which the human intellect cannot but make, when it considers disconnected units of motion instead of continuous motion.

"In the investigation of the laws of historical motion precisely the same mistake arises.

The progress of humanity, arising from an innumerable multitude of individual wills, is continuous in its motion.

". . . Only by assuming an infinitely small unit for observation—a differential of history—that is, the homogeneous tendencies of men, and arriving at the integral calculus (that is, taking the sum of those infinitesimal quantities), can we hope to arrive at the laws of history."—*Leo Tolstoy, in* War and Peace

Mathematics, the schools, and the ORACLE[1]

A. S. HOUSEHOLDER, *Oak Ridge National Laboratory, Oak Ridge, Tennessee.*

It's true, we have machines to do our computing, but, the author asks, "Who is to understand what the machine is doing?"

MY APPEARANCE HERE today to speak to you is in a way fortuitous, not to say presumptuous. Mr. Bass, your program chairman, wrote to me asking for my help in finding a speaker. However, the time was very short. Since I am lethargic by disposition, and inclined toward procrastination, the easiest thing at the moment seemed to be to offer to speak myself. This is how I come to be here.

However, this is not meant to be an apology. I welcome the opportunity to speak to teachers of mathematics, because I believe that what I can say can be of interest to you and may be helpful. Mathematics is my vocation. I might add that it is also my avocation. Formerly I was engaged in the teaching of it, and now I am engaged in the practice of it. There was a time, when I was somewhat younger, when in order to make a living by mathematics alone one had to teach, and although many professions demanded some knowledge of mathematics, there were very few openings outside the teaching profession for mathematicians as such.

Today the situation is quite different. As an indication of the change, consider the personal items in a recent issue of the *American Mathematical Monthly*. On one of the several pages there are items concerning 17 individuals. The following is a list of the professional connections, other than academic, of these individuals: The Goodyear Aircraft Corporation, Sperry Gyroscope Company, Oak Ridge National Laboratory, the Airforce Cambridge Research Center, Convair, Fairchild Aircraft Division, Naval Ordnance Test Center, New England Mutual Life Insurance Company, White Sands Proving Ground and the Ballistic Research Laboratories. My only reason for making the list from this page rather than some others was that it named Oak Ridge. Otherwise there is nothing unusual about it. The names are in alphabetical order, ranging from N to Y, on this page. As you doubtless know, the *American Mathematical Monthly* is the official journal of the Mathematical Association of America, an organization devoted to the interests of collegiate mathematics.

There is a difference, of course, between the utilization of mathematics, and the utilization of the services of mathematicians. The extent to which non-mathematicians must draw upon their mathematical training is not easy to measure. But the list of organizations I have just given provides some indication of the extent to which there is need for the services of mathematicians in the conduct of our society today.

There are those to whom this state of affairs may appear to be a weird paradox, representing at best (or rather, in their eyes, at worst) a stage of transition in our technological development. A news item from the local press in Oak Ridge illustrates what I mean. The item quotes from

[1] Presented to the mathematics teachers of the Middle Section, Tennessee Education Association, Nashville, Tennessee, October 23, 1953.

a statement made by an Oak Ridge citizen in a public meeting. Before reading it I want to assure you that I have not snatched the words out of context. I heard the entire speech, which lasted for perhaps five minutes or so, and the words I shall quote represent a fair summary of what he had to say: "Personality is much more important than grades. . . . The three R's are only tools. Arithmetic? Why they got machines to do all this mathematics!"

Now it just happens that we have at the Oak Ridge National Laboratory, where I work, one of these machines to do all this mathematics. I am in charge of a group of about 20 mathematicians whose responsibilities include the operation of this machine. It is our job, in other words, to get the machine to produce the mathematics. I thought it might be of some interest to you if I try to tell you briefly how it operates, and how we operate. If I am able to give you an intelligible account, perhaps you can then judge for yourselves to what extent we may expect technological unemployment among mathematicians in the near future.

The name of this machine is ORACLE, for Oak Ridge Automatic Computer and Logical Engine. With all due modesty I can say that it is one of the fastest and, for mathematical purposes, one of the most versatile of all machines now in existence. I mention this, not so much to put in a plug for Oak Ridge as to make it clear that whatever I may say or imply about the limitations or shortcomings of the ORACLE apply with equal strength, to any other machine you might name.

Just what, then, can the ORACLE do? It can add, subtract, multiply and divide, and essentially that is all. It is true that it can perform a number of simple non-arithmetic operations, which I shall describe shortly, but these non-arithmetic operations, generally spoken of as logical operations, are relatively unimportant in themselves. They are incorporated primarily to expedite the performance of the arithmetic operations.

I think it should not be hard to see why they are necessary. If you use an ordinary desk calculator, every time you wish the machine to perform an operation you must at least push the appropriate operation button; for almost every operation you must also enter at least one number, and generally two; and following many, if not most, of the operations, you must copy down the result. Some times, as when adding a string of products, a result is merely intermediate and can be left in the machine to be further modified or operated upon. But in any event, the operator has at least one thing to do, and usually several, before any operation can be carried out by the machine.

The ORACLE, however, is able to multiply two numbers in about one two-thousandth of a second; it can add in one twenty-thousandth of a second. Obviously such phenomenal speeds would be of no use whatever if the operator had to make any motions at all to set off each operation. This speed is useful only because the machine itself can, in effect, push its own buttons.

This is the first point I want to make, then, that the ORACLE, like all the other so-called electronic brains, is actually a high speed arithmetic machine, and all its non-arithmetic operations are subservient to the arithmetic ones. In saying this I am certainly not intending to belittle what is indeed an amazing achievement of contemporary technology. But one should understand the machine as it is and not be misled by romantics.

Next, it must be clear that though the machine is capable of pushing its own buttons at the proper times, so that all operations required for any given calculation are performed in the right order, nevertheless the machine itself is not endowed with an understanding of when these times are to be, but can only go by the information supplied by the operator. It can remember a long series of instructions. Indeed, it can operate for minutes, or hours, or even indefinitely, if nothing

goes wrong, after a suitable set of instructions have been given it. But the instructions must be given, and the burden falls back upon the human operator to plan in advance every step the machine is to take, to supply the data upon which it is to operate, and to present all this to the machine at the outset.

This is the subject I should like now to discuss with you in some little detail. Imagine that the ORACLE is at your disposal and you have a problem you wish it to solve. What must you do, precisely, in order to get the ORACLE to give you answers to your problem? In a brief lecture I certainly cannot go into all details, so I shall try merely to illustrate by means of a fairly simple but fairly typical computation. Suppose that somewhere in the course of your computation it is necessary to extract a square root. Clearly one does not employ so elaborate a machine merely to extract a square root, and I am supposing only that this is one small part of the over-all computation. How do we plan for the execution of this part?

I shall have to describe very briefly certain features of the machine. It is convenient to distinguish two major constituents, the memory and the arithmetic unit. The arithmetic unit is, of course, that portion of the machine that performs the actual arithmetic operations, and that corresponds most closely to a desk calculator. In this are two registers which correspond to the dials of a desk calculator. Before an arithmetic operation is performed, one of the operands must be present in one of these registers, just as, say, the dividend must appear on one of the dials of a desk machine before the division begins. Likewise, after the operation is completed, the result will appear in one of these registers.

The memory retains all the input data for the problem, all the instructions for solving it, all the intermediate results obtained at one stage and required at a later one, and all the final results. The memory is divided into a number of cells. Each cell can store a set of binary digits, since this machine uses a binary base rather than a decimal one. To be precise, a cell can store 40 binary digits. A set of 40 binary digits is called a word, and a word may represent a number expressed in the base 2, or it may represent a pair of commands. The cells are numbered in sequence beginning with 0, and the number of a given cell is called its address. The address of a cell is always the same and must, of course, be distinguished from the contents of the cell, that is, the word currently stored in the cell, and which may change many times as the computation proceeds.

As the machine operates it normally goes from one cell to the next, interpreting the contents of each cell when it gets there as representing a pair of commands; it performs these commands and passes to the next cell. However, it is possible to give a special command, called a transfer, which causes the machine to go next to some specified cell instead of proceeding to the next cell as usual. In fact, one can give a conditional transfer command which causes the machine to interrupt the normal sequence only if a certain condition is fulfilled. This makes it possible to cause the machine to repeat a given sequence of operations any specified number of times. This is an extremely important feature, and without it the great speed would be of very little use and might even be a handicap.

Finally it should be explained that the machine, in effect, thinks that all numbers lie between $+1$ and -1. Suppose the number $\frac{1}{2}$ is in cell number 10, and $\frac{3}{4}$ is in cell number 11. Suppose you ask the machine to add the numbers in these cells 10 and 11 and then to put the result in cell number 12. There is no machine word for $1\frac{1}{4}$, which is the true sum, and cell number 12 will receive instead the word for $-\frac{3}{4}$. And if you ask the machine to divide the number in cell number 10 into the number in cell number 11, and to put the result in cell number 13, the machine will go through certain operations and put a num-

ber into cell number 13, but it will not be the true quotient $1\frac{1}{2}$. When you are planning a computation for the ORACLE, therefore, you must take care that everything is properly scaled and that all results will lie on the proper range.

With this understanding, we can get on with the problem. There is a number which we may call a, which the machine will have computed at some stage of your computation, and you are going to need the square root of this number. If the computation has been properly planned, then the machine will have placed the number a in a definite, prearranged, cell in the memory, and you will know the address of the cell. To be specific, suppose a has been placed in cell number 100. You therefore wish the machine at the appropriate stage of the computation to find the square root of the number which will at that time be stored in cell number 100.

Of course, the machine knows nothing about square roots. It understands only addition, subtraction, multiplication and division. It is up to you to devise a sequence of operations of this type that will lead to the production of the square root of a, or of a sufficiently close approximation thereto. And this sequence must be set forth in complete detail in the language the machine understands.

There are several ways of calculating square roots, but the simplest and best from all angles seems to be the use of Newton's method applied to the equation $x^2-a=0$. It is a method of successive approximation. If x_0 is some initial approximation, not too awfully far off, then $(x_0+a/x_0)/2$ is a better one. We can call this x_1, do the same thing with x_1 as was done with x_0, and come out with an approximation that is still better. For example, suppose $a=0.25$. Now you and I know the answer to this at the start, but the machine does not. Suppose $x_0=1$. Then $x_1=(1+a)/2=0.625$, $x_2=0.5125$, and $x_3=0.5001$. Although the initial approximation was quite poor, in three steps the error is reduced to 1 in the fourth place.

Now you can, of course, be sure that the number a to be found in cell 100 is less than 1, and hence will have a square root that is less than 1. One can prove mathematically then that if $x_0=1$, then every x_i in the sequence as defined mathematically will be less than the one which came before, and will be greater than the true square root of a. Furthermore, by continuing long enough one can obtain an x_i that differs from the true square root by less than any preassigned positive quantity. In other words, though one will never find an x_i, in the sequence as defined mathematically, that is exactly equal to the true square root, nevertheless one can find an x_i that deviates by as little as one may choose.

These statements are rigorously correct for the sequence as defined mathematically by Newton's method. But we must now note the fact that in general the machine is not able to give us quite the same sequence. The trouble is that the true quotient of a divided by x_i may, and usually does, require infinitely many digits, and these would require an infinite time to obtain. But the machine can represent only a finite number of digits and we have only a finite time at our disposal. Hence each quotient, and therefore each term in the sequence, will differ slightly from the quantities that are defined mathematically.

The questions then arise: How many properties of the mathematical sequence will be possessed by the numerical sequence actually obtained by the machine, and, in particular, how close will the numerical sequence take us to the true square root? Clearly there is a limit to the attainable accuracy. For if the true square root is not rational, it is certainly not expressible exactly in any finite number of places. There is, however, a unique number that is expressible in the number of places carried by the machine, and that is closer to the true square root than any other such. Is it possible that this number will occur as a term in the numerical sequence? And if so can the machine recognize it and be made to stop on that term?

These questions are by no means merely academic, and their answers are by no means obvious. The fact of the matter is, that although the mathematical sequence is perfectly well defined, I have not yet given a proper and unequivocal definition of the numerical sequence. There are several ways that one might define the numerical sequence, and it turns out that these ways do not always lead to quite the same sequence.

Consider the mathematical sequence once more. Given any term s_i, one forms the next term by the formula $(x_i+a/x_i)/2$. This suggests that one divide a by x_i, add x_i to the result, and then take half of the sum. But remember that our numbers in the machine must be kept less than 1, whereas if a is close to 1, then x_i+a/x_i will certainly be greater than 1. This will not do.

So we might consider next dividing each term by 2 before we add, instead of adding and then dividing the sum. This is a possible way to do it, but the extra division has two slight disadvantages. One is that it requires extra machine operations and correspondingly more machine instructions; the other is that we introduce an extra small error in rounding off after the division. Though not critical, the objections invite further consideration.

You recall that the mathematical sequence is always decreasing. The numerical sequence cannot decrease forever, but one might expect it to decrease at the start. It seems reasonable to suppose that when at some stage the numerical sequence has ceased to decrease, then we have probably come as close as the machine will take us to the true value. This suggests that we calculate the decrement at each step, subtracting this from the previous term so long as it keeps the same sign, and stopping when the decrement vanishes or changes sign. Now the decrement turns out to be $(x_i-a/x_i)/2$. Thus, in the previous example, from $x_i=0.625$ we would have the machine calculate the decrement $\Delta(x_1)=0.1125$ and then obtain $x_2=0.625-0.1125=0.5125$. Since x_i and a/x_i are both less than 1, their difference is also less than 1, and we avoid both the illegitimate addition and the extra division.

Even now, however, the numerical sequence is not completely defined. We could divide a by x_i and subtract the quotient from x_i; or we could divide $-a$ by x_i and add the quotient to x_i. In either case we would take half of the result as the required decrement. With either scheme we would introduce a conditional transfer instructing the machine to subtract this decrement from x_i and repeat, if the decrement is positive, but otherwise to go on to something else since it has then found as close an approximation as it can get. A third possibility, not apt to suggest itself immediately, is to divide $-a$ by x_i, subtract from $-x_i$, and then divide by 2. This gives the negative of the decrement and is to be subtracted from x_i if negative. Even this does not exhaust the list of possibilities, but perhaps it is sufficient for illustrating the point. When one of these three possible modes of procedure is selected, then, and only then, is the numerical sequence completely defined.

It turns out that these three procedures actually define three different sequences. The third one is somewhat preferable to the first and either is preferable to the second. When the third one is used, the decrement never changes sign but ultimately vanishes. When this happens the approximation is for most values of a the best possible.

We have now selected a precise sequence of arithmetic operations for the machine to perform in order to find the square root. The rest is routine. We decide in what specific memory cell we wish to send the final approximation x, and in what part of the memory we shall store the instructions for doing the computation. Then we write these down, one after the other, in the language of the machine. It takes 16 distinct commands, or 8 words. Two of these commands are transfers. One transfer command tells the machine where the instructions for the next task are to be found after the decrement has become equal to zero. The other one tells the ma-

chine to go back to the beginning and repeat the cycle when the decrement is not yet zero. In the worst case the machine will go through the cycle 39 times before it finishes.

I have chosen the square root for my illustration because it is quite simple, but nevertheless fairly typical. Whether it is the 16 commands for finding a square root, or the several hundred commands for inverting a matrix, or the one or two thousand commands that might be required for finding the numerical solution of a system of differential equations, they must be explicit to the last detail, for the machine has no imagination whatever, and they must be based upon a careful analysis of the problem at hand.

In one respect the square root problem is extremely atypical. We can say definitely that the answer is at worst second best, and we can lay down very narrow limits of error. Generally, however, the errors due to rounding off the results of multiplication and division build up in a manner that is difficult to analyze so that our estimates of them are only very crude. Sometimes even the best schemes we have been able to devise still leave us with results that may be not even second or third best, at least without making the sequence of instructions far more elaborate than is ordinarily worth while.

In this technological age, machines have created for us material comforts undreamed of in times past, and they have relieved us of much drudgery, both physical and, even, mental. But no machine yet conceived can provide understanding. If the ORACLE's performance deviates in even the slightest respect from what you expect of it, then either you have unwittingly given it incomplete or improper instructions, or else, what is possible but much less likely, something has gone wrong with the machine. Only you, and not the machine, can understand what is needed, and how the machine can get it for you. This is the point I have been trying to make, perhaps laboriously, by my example of the square root.

And so, if we are told that inasmuch as there are computing machines, or thinking machines, or electronic brains, or what have you, therefore it is no longer important that students learn mathematics, we can be assured that nothing can be further from the truth. Millions of dollars are being spent every year by industry and government on the design and construction of these machines. Here again is evidence of the importance of the role played by mathematics in the operation of contemporary technology. But only the mathematical understanding of human beings can recognize and formulate the problems that require solution, can devise the methods to be used for solving them, and can interpret and apply the solutions once they are found. Our friend, whom I quoted at the outset, is laboring under a misconception that is shocking and extremely dangerous. It would undermine the foundations of the very technology that can produce and effectively utilize these machines for us to relax and expect them to relieve us of the necessity to learn and understand mathematics.

With regard to personality, which our friend considers so important, I might remark that I, too, consider it important, especially in a group such as ours which requires much teamwork. But while I would not hire someone whose personality did not strike me as reasonably good, I would not even bother to interview any person whose grades were not well above average.

Let me conclude with another brief quotation. You recall the Polish UN delegate who recently escaped from his hotel in New York. The newspapers quote him as saying that Soviet grand strategy is based upon "the progressive destruction of the cultural, economic, and political foundations of the free world." Among the cultural and economic foundations of the free world, let no one underrate the importance of mathematics. The obvious place to attempt to undermine these foundations is in the schools. It is up to us whether we will withstand the attack or assist it.

The new role of mathematics in education[1]

PAUL BROCK, *Manager, Technical Services Department, ElectroData Corporation, Pasadena, California.*

The author points out industries' desperate need for personnel trained in mathematics, and he argues that the scarcity of trained personnel is due, in part, to reduced mathematical requirements in the high schools.

TODAY THERE IS an urgent need for mathematically trained personnel in industry. This need is increasing and is expected to continue increasing for many years to come. The principal sources of individuals to fill this need are, of necessity, the schools of the country.

But the number of people with mathematical training that the schools are turning out is insufficient. Even more critical is the number of capable students that have been graduated. Although there is the need for people with mathematical background, industry is not interested in technically weak individuals. In most cases, these are worse than no help at all.

The principal reason for this bad situation is, I feel, that mathematics teachers have in recent years been severely on the defensive at all academic levels. Changing concepts of education have also worked to the detriment of mathematics in the school curricula.

The classical educational theory that the schools should produce well-rounded, academically trained individuals, has been virtually abandoned. In its place the schools are required to turn out well-rounded, socially adjusted individuals who can ably fulfill the duties of citizens and be a positive asset to the community. The aims of this newer theory are well justified. The effect on mathematics training, however, has been that the arguments relating to mental discipline, development of logical thinking and the traditionally weak arguments for applications of mathematics have become insufficient to overcome the much stronger arguments for training in social science, economics and the arts as a background for intelligent citizenry. Thus, mathematical requirements have been gradually reduced in the shifting balance of subject requirements for the school population.

There are other reasons for the weakening of formal mathematics training:

1) There has been a lack of sufficient mathematical guidance for students at all levels.

2) Mathematics teachers throughout the country are not as fully aware as they should be of recent trends in their subject.

3) The very need for mathematicians has produced a detrimental effect on mathematics training in that the relatively large number of students entering industry has decreased the numerical quantity and academic quality of individuals entering the teaching profession.

4) There is a cumulative environmental effect of several generations of students that have been produced during this period

[1] Presented at the annual meeting of the California Conference of Teachers of Mathematics, University of California, Los Angeles, July 9, 1954.

of decline. The average home today is not interested in the mathematics program of the schools nor do parents tend to encourage their children towards stronger mathematical training programs.

I should like to point out that modern trends indicate that there is no reason for mathematics teachers to have a defensive attitude. Mathematics teachers have more than enough ammunition to conduct a strong positive campaign, and further to enlist the support of other departments for the inclusion of a stronger, more intensive and more universal mathematics program throughout the school system.

The changing educational philosophy that has weakened mathematics in the curriculum, today can be used to strengthen the subject in the schools.

Mathematics training is vital and essential to understand the impact of modern technology and science on our social and political economy.

Mathematicians and other scientists have found that the use of mathematics has been tremendously expanded in both pure and applied scientific work and in actuarial and statistical work. More mathematics teachers are needed now than ever before.

The running of our government today requires detailed knowledge of military defense, nuclear energy, public power, statistical techniques, etc. Public officials should know about the nature of nuclear fission, the characteristics of a radar net, the difficulties to be overcome in the construction of supersonic jet missiles and innumerable other topics of this type. How can a key public figure make decisions on questions involving these subjects without basic mathematical knowledge? He does not need a complete mathematical training, but certainly he should be cognizant of certain concepts such as literal formulas, rates of change, stability, and standard and nonstandard geometrical configurations. Are not these and similar topics the very heart of our basic mathematical curriculum?

The industrialist today must obviously have a similar type of training if his interests include manufacturing, research or development of key items of our technology. But more generally, important new techniques of management itself are strongly rooted in mathematical concepts. The field of economics has been revolutionized by modern ideas of econometrics coupled with advances in computing techniques. Basic tools used in modern econometrics are matrix theory, integral equations and functional analysis.

In relation to reduction of overhead by management, it has been said that the one large unexploited field is that of automatising office operations. This involves the installation of complex mathematically designed equipment. Is every office manager going to be a mathematician? Or, should office managers have a strong mathematical background to understand the equipment they are using?

I could continue in this vein to describe automatic process control, optimization of production scheduling, the expanded use of statistical feedback by management and numerous other topics, but I think the point is quite clear. The students today who will soon run our government, operate our public institutions and direct our business and industry must have a strong mathematics background.

The lay public is perhaps the most important segment of our population to feel the effects of these changes. They are the bulk of our citizenry. They must understand what their representatives tell them. They must express their opinions to their representatives on questions which, in essence, are technical. Unless we are to assume that our traditional concepts of a popular government are completely meaningless, we must prepare our citizenry to have sufficient general information to understand the basic problems that confront our government today. This meansa strong program of elementary mathematics.

Finally, I would like to point out that though the relevancy of mathematics to

the understanding of questions normally under the aegis of other departments is a problem of these other departments, I am not at all sure that the faculties of these other departments understand the technicalities of the questions involved. You must instruct your colleagues in the modern role that mathematics plays in their own fields. When they understand, they must demand a stronger mathematics program to support their own fields.

But first, the teachers of mathematics must instruct themselves and make positive efforts to obtain a comprehensive background in the ramifications of these recent mathematical applications. This effort must be made by teachers at all levels so they will be able to strengthen there mathematics program most effectively. Perhaps more important, they must implement vital orientation programs for the students and the students' homes.

Mathematics is the key to a comprehension of our modern complex, technological society. The mathematics teachers are in the critical position of seeing that the basic information for this understanding is disseminated to our rising generation.

What's new?

BOOKS

SECONDARY

Trigonometry with Tables, A. M. Welchons and W. R. Krickenberger, Chicago, Ginn and Company, 1954. Cloth, vii+419 pp., $3.20.

COLLEGE

Algebra for College Students, William M. Whyburn and Paul H. Daus, New York, Prentice-Hall, Inc., 1955. Cloth, xi+290 pp., $4.25.

Arithmetic, For Teacher-Training Classes (4th Edition), E. H. Taylor and C. N. Mills, New York, Henry Holt and Company, 1955. Cloth, ix+438 pp., $4.25.

Trigonometry, Roy Dubisch, New York, The Ronald Press Company, 1955. Cloth, xiv+396 pp., $5.00.

MISCELLANEOUS

Mathematics for Technical and Vocational Schools (4th Edition), Samuel Slade and Louis Margolis, New York, John Wiley and Sons, Inc., 1955. Cloth, ix+573 pp., $4.48.

BOOKLETS

Solving Arithmetic Problems Mentally, Bureau of Extension Service, Iowa State Teachers College, Cedar Falls, Iowa. Illustrated booklet written by Jack V. Hall; 33 pp., 25¢ (send remittance with order).

Tear Sheets for Teaching, The Visual Instruction Bureau, Division of Extension, The University of Texas, Austin 12, Texas. Illustrated booklet written by Charles H. Dent, Leonard B. Ambos, and Nancy M. Holland; 24 pp., $1.00.

Your Career with the Instrument and Control Industry, Hampton M. Auld, Executive Secretary, Recorder-Controller Section, Scientific Apparatus Makers Association, 522 Fifth Avenue, New York 36, New York. 6 page pamphlet, free.

DEVICE

Radian and Circle Demonstrator (Cat. #7501), W. M. Welch Scientific Company, 1515 Sedgwick Street, Chicago 10, Illinois. 21" disk made of composition-board with cord attached, $7.50.

EQUIPMENT

Logarithm and Trigonometric Functions Chart (Cat. #7550), W. M. Welch Scientific Company, 1515 Sedgwick Street, Chicago 10, Illinois. Reversible chart, 76"×52" roller and hardware included, $15.00.

TESTS

Measuring Power in Arithmetic, Silver Burdett Company, 45 East 17th Street, New York 3, New York. Achievement tests with norms for grades 3–8 by Robert Lee Morton with contributing consultant, D. Banks Wilburn; package of 35, $3.50 (net), one complete test, $.15 (net).

Progress Tests, Silver Burdett Company, 45 East 17th Street, New York 3, New York. Achievement tests based on the *MAKING SURE OF ARITHMETIC* textbook program, grades 3–8; package of 35 sets, $5.00 (net), $.20 (net) per set.

Transforming the law of cosines for computational purposes

BENJAMIN GREENBERG, *Ramaz High School, New York City.*

The law of cosines has always frustrated those teachers who wish to use it computationally. The transformation developed in this paper makes it more amenable to the use of logarithmic computation.

A FREQUENT CRITICISM of the law of cosines is that it is not convenient for logarithmic or slide rule computation. This deficiency may be overcome by a method to be developed in this article. We know that

(1) $a^2=b^2+c^2-2bc\cos A$

(2) $\cos^2\dfrac{A}{2}+\sin^2\dfrac{A}{2}=1,$

and, since

(3) $\cos^2 A-\sin^2 A=\cos 2A$

it follows that

$$\cos A=\cos^2\frac{A}{2}-\sin^2\frac{A}{2}$$

Equation (1) may now be written in the following form:

$$a^2=(b^2+c^2)\left(\cos^2\frac{A}{2}+\sin^2\frac{A}{2}\right)$$
$$-2bc\left(\cos^2\frac{A}{2}-\sin^2\frac{A}{2}\right)$$

or

(4) $a^2=(b-c)^2\cos^2\dfrac{A}{2}+(b+c)^2\sin^2\dfrac{A}{2}.$

Equation (4) is in the form $x^2=y^2+z^2$ which suggests drawing an auxilary right triangle as in Figure 1 and incorporating the data given in equation (4).

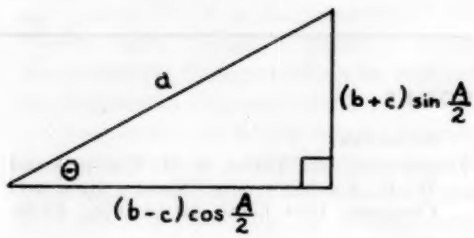

Figure 1

From the diagram

(5) $\operatorname{Tan}\theta=\dfrac{(b+c)\sin\dfrac{A}{2}}{(b-c)\cos\dfrac{A}{2}}=\dfrac{b+c}{b-c}\operatorname{Tan}\dfrac{A}{2}.$

The angle θ can be determined conveniently by use of logarithms from equation (5).

From the auxilary triangle

$$\sec\theta=\frac{a}{(b-c)\cos\dfrac{A}{2}}$$

so that

(6) $a=(b-c)\cos\dfrac{A}{2}\sec\theta.$

Equation (6) is suited for the logarithmic computation of a. Illustrative example: In the triangle ABC,

$$b = 37.2$$

$$c = 22.3 \quad \text{Find } a$$

$$A = 29°38'$$

Using formula (5)

$$\text{Tan}\,\theta = \frac{b+c}{b-c} \tan \frac{A}{2}.$$

$$\text{Log tan}\,\theta = \log(b+c) + \log\tan\frac{A}{2}$$

$$-\log(b-c)$$

$b+c=59.5, \quad \log(b+c)=1.7745$

$b-c=14.9, \quad \log(b-c)=1.1732$

$\frac{A}{2}=14°49', \quad \log\tan\frac{A}{2}=9.4225-10$

$$\text{Log tan}\,\theta = 10.0238-10; \quad \theta = 46°34'$$

From (6) it follows that

$$\log a = \log(b-c) + \log\cos\frac{A}{2} + \log\sec\theta$$

and, after substituting the appropriate values,

$$\log a = 1.3212; \quad a = 20.95$$

Both the law of cosines and the law of tangents could have been used to solve the above example. It is a well known fact that the law of cosines method is very cumbersome when the sides of a triangle are given to three or more significant digits. The law of tangents is convenient for finding side a in the illustrative example above; but it does not find a directly. We must first find either angle B or angle C and then find side a by the law of sines.

The law of cosines may be readily used in a problem similar to the illustrative example when numbers of two significant digits are used. It is not convenient to use logarithms in such a solution.

The reader should note that if $c>b$, then formula (4) may be written

$$a^2 = (c-b)^2 \cos^2\frac{A}{2} + (c+b)^2 \sin^2\frac{A}{2}$$

and in the auxilary right triangle we employ $c-b$ in place of $b-c$. It will make no difference how the symbols b and c are assigned to the data, since $b+c=c+b$ and $bc=cb$. Notice also that since

$$A < 180°, \qquad \frac{A}{2} < 90°,$$

and hence

$$\sin\frac{A}{2} \text{ and } \cos\frac{A}{2}$$

are both greater than zero.

Have you read?

Hartung, Maurice L. "Modern Methods and Current Criticisms of Mathematical Education," *School Science and Mathematics*, February 1955, pp. 85–90.

The statements of those who criticize mathematics and modern education in general are familiar to you. Did you ever feel that we were being criticized because we are unwilling to accept modern education? This is the thesis of Dr. Hartung and he defends it very well. Questions that should be considered are: Have we provided for individual differences? Have we included in the curriculum the points of view of modern mathematics? Have we used our time efficiently by reorganization of content and careful selection of those aspects of mathematics which have proved most valuable? Have we put into effect those improved practices so well laid down by Klein, Moore, and others?

For the teacher who is looking ahead to make a contribution to mathematics teaching, this article is a must. It is one of the most thought-provoking I have read in a long time.

Langman, Harry. "A Problem in Checkers," *Scripta Mathematica*, vol. XX, September-December 1954, pp. 206–208.

All boys and most girls have at some time been enthusiasts of the checker board. Here is discussed a game of solitaire with checkers which they say is hard to win. Arrange your checkers in two rows around the outside of your board. Every move is a jump and every jump removes a checker. Only diagonal jumps are allowed. Do you think you can end the game with only one checker on the board? Your better students will be intrigued with this article and the mathematics included.—Philip Peak, *Indiana University, Bloomington, Indiana*

On explanation

KENNETH B. HENDERSON, *College of Education, University of Illinois, Urbana, Illinois.*

A note on "scientific explanation" with suggestions as to how it can be taught to high school students.

ALMOST ALL mathematics teachers say that they explain in the course of their teaching. One way to find out what they mean by the term is to listen to what they say when they explain. Another way is to read and listen to what they accept as explanations from their students. When both of these are done, it is evident that there are three general senses in which the word "explain" is used in the classroom. The purpose of this article is to explicate this concept and to offer some suggestions for practice.

One of the meanings of the term "explain" is illustrated when a teacher says, "Now if you all will listen carefully, I shall explain how to bisect a line segment." The teacher then describes the steps that should be followed. Another instance of this meaning occurs when a teacher asks a student to explain how he solved the problem he has written on the board. The student reads what he has written there. The teacher asks if any of the class have questions. Once these are answered, if there are any, the teacher passes to the next problem or part of the lesson. From this we may infer that he is satisfied with this "explanation."

A second meaning of "explain" is illustrated when a teacher asks, "Explain what the word 'trapezoid' means." A student gives the correct definition of the term, and the teacher is satisfied. Evidently this was the sense in which he was using the term "explain" in this context.

A third meaning of "explain" is illustrated when a teacher says, "The laws of exponents apply equally to logarithms because logarithms are exponents." It is also illustrated when a teacher asks a student how he can pass from the equation $x+10=24$ to the equation $x=14$. The student says, "Equals subtracted from equals leave equal remainders. I subtracted 10 from each side of the first equation. Therefore, $x=14$." The teacher's acceptance of this reasoning indicates what "explain" means to him in this context.

It is obvious that these three senses in which "explanation" is used are not the same. True, all three have the same general function, namely, to make something clear or clearer so the students can understand it. But they accomplish this function in different ways. When the teacher uses "explain" in the first sense, he clarifies by description or narration. When he uses the term in the second sense he clarifies by interpretation; he gives the meaning of a term or phrase. When he uses the term in the third sense, he clarifies by deduction; he points out a relationship.

When "explanation" is used in the third sense, it is often called scientific explanation.[1] From a pedagogical standpoint it is probably in this sense that the term should be used in the classroom. For this reason, let us consider the nature of this sense in which the term is used.

[1] See Hans Reichenbach, *The Rise of Scientific Philosophy* (Berkeley: The University of California Press, 1951), Ch. 2.; Arthur Pap, *Elements of Analytic Philosophy*, Ch. 11; Norman Campbell, *What Is Science* (New York: Dover Publications, 1952), Ch. 5.

The nature of scientific explanation

We explain phenomena, relationships, operations, and assertions. In the mathematics class explanation concerns chiefly the last three. But whatever is explained, the form of the explanation is the same. We first describe the phenomenon, relationship, operation, or assertion by a proposition. Then we select two premises from which the proposition to be explained follows as a necessary conclusion. One of these premises will always be a proposition of greater generality than the proposition to be explained. The other premise subsumes the proposition to be explained under the generalization (first premise). For example, to explain why $\frac{6}{8}$ can be reduced to $\frac{3}{4}$ we might say. "The numerator and denominator of a fraction can be divided by the same number except zero without changing the value of the fraction. Both the numerator and denominator of $\frac{6}{8}$ were divided by 2 thereby obtaining $\frac{3}{4}$. Therefore, $\frac{3}{4}$ is equivalent to $\frac{6}{8}$."

If both premises are stated as in the explanation above, the explanation is said to be a complete explanation. If either (but of course not both) premise is suppressed, the explanation is said to be incomplete.

Most of the explanations we give, read, and hear are incomplete. A teacher says, "You cannot write the sum of $5a$ and $5b$ as one term because the two terms are not alike." In this case the major premise—the generalization or principle—is not stated.

The teacher might have said, "You cannot write the sum of $5a$ and $5b$ as one term because only like terms can be combined into one term by adding." This explanation suppresses the minor premise, the proposition that indicates that the particular operation is an instance of a general principle.

Teleological explanations

When applications of mathematics are made in human affairs, there is the possibility of a teleological as well as a scientific explanation. In a plane geometry textbook the question is asked, "When men set fence posts, why do they first set the end posts?"[2] A student might say, "Because the men want the fence to be set accurately." By "accurately" we find that the student means a small deviation from a previously determined straight line. His explanation might be completed as follows:

Men who set fences want to set them accurately.

Setting the end posts first will result in greater accuracy than setting any other two posts first.

Therefore, men set the end posts first.

A second student, in answering the question, might say, "So the men can save time." His complete explanation might be:

Ordinarily, men want to save time.

Setting the end posts first saves time.

Therefore, the men set the end posts first.

Both of these explanations are teleological. The minor premise in each is an empirical generalization which emerges from a study of people's purposes and values.

Explanations given in class, of course, should be true explanations. "True" in the context of mathematics means based on axioms, postulates, definitions, or theorems. The test of an adequate explanation cannot be only logical validity. It is not too hard to give possible explanations; that is, pairs of propositions from which the proposition to be explained can be deduced. For example, we might "explain" why, if the left side of $2y=10$ is divided by 2 the right side also must be divided by 2, by saying that this is the fair thing to do. We could complete the explanation, and make the inference necessary. But the "explanation" still remains a pseudo-explanation.

Implications for classroom teaching

It is well known that one of the hardest jobs in teaching is communicating with

[2] A. M. Welchons and W. R. Krickenberger *Plane Geometry* (Boston: Ginn and Company, 1951), p. 9.

the students—selecting words or other symbols and arranging them in some sequence so the students will understand. Students find words which have a multiordinal quality difficult. They must attempt to select the particular meaning of the word from the context in which the word is used. And often the context does not provide sufficient cues. Consider the student who is asked by the teacher to explain how he solved a problem. He will find it hard to tell from the context whether the teacher wants him to describe how he solved the problem or to state the mathematical relationships involved.

Mathematical ideas are difficult enough as they are. It would seem reasonable for teachers to be particularly careful in their diction so that they do not unnecessarily increase the difficulty by poor communication. Specifically, it seems just as easy to say, "Tell us how you worked the problem" instead of couching the statement in terms of "explain." And it seems just as easy to say, "What does the term 'trapezoid' mean" rather than to phrase the question in terms of "explain." "Explain" then can be reserved for the sense of scientific explanation. This should make it easier for students to know what is expected of them.

Hand in hand with this increase of precision in the use of words should be the development of the concept of explanation; that is, the nature of an explanation. Explanations students give in class may be used as starting points. The teacher can tell the students the salient features of an explanation and illustrate his points by using mathematical propositions. As practice exercises, students may be asked to state the proposition (terminology suitable to the maturity of the students should, of course, be used) needed to complete an explanation in which only one reason is given. Some exercises would require the principle (theorem, rule, definition, assumption); others would require the statement indicating the relevance of the case in point to the principle.

As a second exercise, students might be asked to identify the reason they give for the explanation as either the principle or a statement making the thing to be explained a particular instance of the principle.

A third kind of exercise might consist in having students distinguish between explanations and various pseudo-explanations.

In a project in the improvement of thinking carried on in Calumet High School in Chicago,[3] we found exercises like the following effective in teaching students what is meant by an explanation.

Exercise on explanations

Part I. A dog is tied to a pole by means of a chain. The chain is attached to the pole by a swivel. The dog walks completely around the pole keeping the chain stretched out. Explain why his path is a circle.

Part II. Check the statement below which *best* explains why the locus is a circle.

____1. The dog walked completely around the pole.
____2. The locus of points around a fixed point is a circle.
____3. The dog wanted to stay as far away from the pole as he could.
____4. The dog walked around the pole in a circle because he did not like to be tied up.
____5. The locus of points at a fixed distance from a fixed point is a circle.

Part III. You must have had a reason for choosing the statement you checked. This is assuming that you did not guess. Check the *one* of the following statements which tells why you checked the explanation in Part II. If none of these statements tells why you checked the statement, write your reason at the end of the list.

____ 1. The statement I checked was the reason.
____ 2. The question probably has something to do with geometry, and the statement I checked is a geometry theorem.
____ 3. To explain a fact, find another fact related to the first fact.
____ 4. A statement is explained when it is shown to be a particular instance of a general law or theorem. The statement I checked is a general law or theorem.
____ 5. A statement is explained when it is

[3] Mathematics teachers involved were Miss Irene McEnroe who taught a class in intermediate algebra and Mrs. Florence Graham who taught a class in plane geometry.

shown to be a particular instance of a general law or theorem. The statement I checked showed that the statement about the dog's path is a particular instance.

____ 6. I put myself in the dog's place.

____ 7. Any animal that is tied tries to get away, but the rope that ties it prevents it from getting away.

____ 8. Any animal that is tied is unhappy so it walks around.

____ 9. The statement contains the word "because," and this word is used in explanations.

____10. ________________________________

We considered that choice 5 in Part II was correct and choice 4 in Part III was correct. We secured the false choices by having students previously write several explanations discursively. We identified the common misunderstandings about explanation. Then in building exercises like the one given, we phrased the false choices so as to allow the misunderstandings to be demonstrated. The teachers corrected the misunderstandings in the course of the class discussion of the exercises.

Exercises like these were designed for both the plane geometry class and the intermediate algebra class. The following is an exercise used in the algebra class. In this exercise choice 5 in Part II and choice 4 in Part III were considered correct.

Exercise on explanations

Part I. If $\log 3 = 0.4771$ and $\log 5 = 0.6990$, $\log 15 = 1.1761$.

Part II. Check the statement below which *best* explains why the statement in Part I is true.

____1. The characteristic of 15 is one less than the number of digits to the left of the decimal point.

____2. The table of logarithms gives 0.1761 as the mantissa of 15, and the characteristic of 15 is 1.

____3. The logarithm of a number is the power to which a number called the base must be raised to equal the given number.

____4. $\text{Log } 15 = 1.1761$ because $\log 3 = 0.4771$ and $\log 5 = 0.6990$.

____5. $\text{Log } x + \log y = \log xy$.

____6. Add the logarithms of 5 and 3.

Part III. You must have had a reason for choosing the statement you checked. This is assuming that you did not guess. Check the *one* of the following statements which tells why you checked the explanation in Part II. If none of the statements tells why you checked the statement in Part II, write your reason in the lines at the end of the list.

____1. It described how you get the logarithm of 15.

____2. It was the only statement that was correct.

____3. The statement contained the word "because," and this word is used in explanations.

____4. A statement is explained when it is shown to be a particular instance of a general law. The statement I checked is a general law.

____5. A statement is explained when it is shown to be a particular instance of a general law. The statement I checked shows that the statement in Part I is a particular instance.

____6. The statement can be proved to be true.

____7. ________________________________

Once the concept of an explanation has been developed, the teacher should insist that the students use the term and its derivatives correctly.

An understanding of what it means to explain something provides a student with a way of relating his knowledge. A particular proposition can be seen as an instance of a more general proposition which explains it. The more general proposition may itself be seen as a particular instance of an even more general proposition. Thus the student pyramids his generalizations. This is what gives him power in thinking and increased ability to solve problems.

"Upon the subject of education . . . I can only say that I view it as the most important subject which we as a people can be engaged in."—*Abraham Lincoln*

"We have faith in education as the foundation of democratic government."—*Franklin D. Roosevelt.*

The teaching of mathematics to the blind[1]

C. M. WITCHER, *Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts.*

Those who have sight will always marvel at how the blind learn mathematics and all will be glad to know how recent improvements in the Braille system facilitates the study of mathematics by the blind.

Introduction

In starting to think about this subject, I could not help being reminded of a personal experience that was rather distressing at the time, but which has since struck me many times as quite amusing. It happened during my first day in freshman analytic geometry class. The professor, who was a very enthusiastic devotee of his subject, had doubtless completely forgotten after the first few minutes of class that one of his students was blind. He was traditionally famous for using chalk of many different colors in his discussions, and so it is certainly not surprising that he should have emphatically declaimed the following: "Now I want you to note this *carefully*, fellows. By definition, the *slope* of this line is the ratio of the blue distance to the red distance, and its Y intercept is the green distance."

Another experience happened to a blind friend of mine while he was taking a quiz in college physics. He was making his numerical calculations in Braille, having frequent occasion to refer to a four-place table of logarithms which he had transcribed into Braille and placed in a standard cardboard ring binder. Suddenly the professor, with considerable accusation in his voice, said: "I'm sorry, John, but you are not allowed to refer to notes in this quiz!" John explained that the material to which he had been referring was a table of logarithms, but the teacher, not being at all familiar with Braille, was entirely unconvinced. He insisted that John should close the book immediately. Finally, in desperation, John brought the book to the professor, pointing out to him the neat columns, each of which contained four Braille characters (the digits of the mantissa), and also that the column on the left contained only two characters (the value of N). Happily, the professor was finally convinced.

Speaking more seriously, the blind mathematics student of today is inconceivably more fortunate than those of Nicholas Saunderson's time (1682–1739). Saunderson, the first of a number of outstanding blind mathematicians, occupied for 30 years the chair of Lucasian Professor of Mathematics at Cambridge University in England, which Newton had previously occupied. In fact, it was at Newton's insistence that Saunderson was appointed to the position. Saunderson devised what was probably the first scheme for mathematical calculation in which the symbols were perceptible by touch. His device consisted of a one-foot-square board ruled off into small squares, in each of which nine small holes were drilled. By the insertion of one or more pins into these holes, the numerals, signs of operation, and the like, could be represented for touch reading. This system is the forerunner of the device

[1] Presented at the Dinner Meeting of the Association of Teachers of Mathematics in New England held at the Massachusetts Institute of Technology, Cambridge, Massachusetts.

known as the Taylor slate, which was widely used in schools for many years, and is even used to some extent today. Saunderson produced an extensive treatise on algebra, as well as a work on Newton's method of fluxions.

I devote this much of our time to Saunderson because he must have been a particularly intriguing individual. Apparently he contrived numerous other devices to aid him in his mathematical work. In speaking of him, Diderot, in his famous "Letter on the Blind," describes four small rectangular boards, the flat sides of each of which were ruled off into nine small rectangles. In each of these rectangles there appears a set of 10 five-digit numbers. One of these sets is shown in Figure 1. To this

94084
24186
41792
54284
63968
71880
78568
84358
89464
94030

Figure 1

day it is not known just what these tables represented. Also, Diderot did not comment on the way in which the digits were inscribed on the boards, whether by Saunderson's pin system or in some other way that would be suitable for touch reading. It is known that Saunderson did considerable work in the theory of numbers, and it may be plausible to surmise that these tables were in some way connected with that branch of mathematics.[2]

Beginning of systematic education

The history of the education of the blind is inseparably bound up with the development of apparatus to take the place of vision. The Frenchman, Valentin Haüy, was probably the first to attempt mass education of the blind. For numerical calculations, he used a set of movable type, bearing ordinary Arabic numerals, together with a board filled with small square holes into which the type could be set. It is almost unbelievable that Haüy's movable-type system continued to be used in some schools for the blind up to a few years ago. Speaking personally, though I never attended a school for the blind, most of my elementary school arithmetic was done with type of this kind. You may easily imagine the time lost after the completion of each calculation in carefully replacing the type in ordered rows, five of which were at each side of the board. After all, this was the only system of which my parents were aware.

It is of considerable interest in studying the development of apparatus for aiding

[2] Shortly after the preceding address was delivered, several of us in this laboratory became curious as to the significance of the table in Figure 1. Yngve made the suggestion that the apparent anomaly in magnitude between the first number in the table and all subsequent entries might be due to the fact that the tabulation represented the last five digits of a function comprised of six-digit numbers, the first digit being omitted, as, for example, in the case of the characteristic and mantissa of a logarithm. The idea that the table might actually consist of a set of logarithms was suggested by Yngve's observation that, when the second number in the table was assigned a characteristic of 1 and the first number a characteristic of 0, their difference was 0.30102. The common logarithm of 2 is 0.30103. Carrying this idea further, Witcher subtracted the first entry successively from all subsequent entries with Yngve's assignment of characteristics. The results are very close to the logarithms of the integers 2-10, as shown in Table A-1, in which the column marked D contains the results of these subtractions.

Table A-1

n	$\log_{10} n$	D
2	0.30103	0.30102
3	0.47712	0.47708
4	0.60206	0.60200
5	0.69897	0.69884
6	0.77815	0.77796
7	0.84510	0.84484
8	0.90309	0.90274
9	0.95424	0.95380
10	1.00000	0.99946

These results definitely convinced us that we were dealing with a set of logarithms, but the fact that all of the entries were even numbers continued to throw us off the right track. We made the obvious guess that the entries must each have been obtained by multiplying a logarithm by 2, and hence that the table represented a set of logarithms of squares. We frequently tried to find correspondences between the entries of Figure 1 and various tabulated functions, but the

the blind to compare that which was conceived by people with normal sight with that which was developed by the blind themselves. Haüy had normal vision. Professor Henry Moyes of Manchester, David MacBeath of Edinburgh, and his student, Lang, were all blind. Perceiving the inefficiency of Haüy's movable type, they proceeded through a succession of developments originating from Saunderson's calculating board. Their final apparatus was slightly refined by the Reverend William Taylor, of York, into what we now know as the Taylor slate. This consisted of a board, or metal frame, filled with octagonal holes. A variety of type was used in which the pieces were in the form of square prisms having on one end a raised line near one of the edges, and on the other end two raised dots similarly placed. Obviously, a piece of type could be inserted into one of the holes in eight different angular orientations, and, since the ends were easily distinguishable by touch, sixteen different symbols could be produced with a given piece of type. In picking up a piece of type from the supply compartment, the blind student's fingers could quickly select the proper end and angular orientation, the whole process requiring not much more time than the hand movement to bring the type to the proper hole in the board and insert it. Type with other sorts of markings on the ends was also developed for use in algebra. At the end of a calculation, the type could be quickly raked back into the supply compartment, so that the time required for calculations was considerably less than that with Haüy's system. Taylor's apparatus was used extensively in schools until Braille writing machines became widely available a few years ago.

Until very recently, the apparatus available for producing geometrical figures, graphs, and other sorts of diagrams has not been too satisfactory. One blind man of the sixteenth century is alleged to have developed a wax pencil that could produce a raised wax line. His method for the preparation of these pencils was, unfortunately, never disclosed, but it is of some interest that efforts to duplicate or improve on his discovery led eventually to the development of the fountain pen. The earliest apparatus used widely in schools for the blind consisted of a shallow tray filled with wax. By drawing upon the surface of the wax, a tactually perceptible groove could be produced. Erasing was done either with a roller or by heating the wax. Obviously, this was not a very satisfactory method, especially because a groove, or depressed line, is much less readily felt than a raised line. Considerable improvement was achieved through the use of a large square board covered with cork or balsa wood, into which pins or tacks could be inserted. To produce straight lines the pins could be connected by rubber bands. Wires, or better, small round solder could be used to produce curves. Much use has been made of the so-called "spur wheel," a very small toothed wheel mounted like a caster on the end of a handle. Wheels of this kind, when properly made, produce very beautiful dotted raised lines; the only difficulty is that they

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idea of squares continued to confuse us. Finally, Washington made the happy, but somewhat deflating, discovery that the table simply represented the values of log sin θ for intervals of 0.5° from 0.5° to 5°, as exhibited in Table A-2. The difference of 2 in the last digit of one of the entries evidently represents an error on the part of Saunderson or one of the subsequent transcribers of his table. What, after all, could be more logical for a blind mathematician who frequently had to perform elementary numerical calculations than to transcribe for his own use a set of trigonometric tables?

TABLE A-2

$\theta°$	$L \sin \theta$	Fig. 1
0.5	7.94084	94084
1	8.24186	24186
1.5	8.41792	41792
2	8.54282	54284
2.5	8.63968	63968
3	8.71880	71880
3.5	8.78568	78568
4	8.84358	84358
4.5	8.89464	89464
5	8.94030	94030

Lamar Washington, Jr.,
Victor Yngve,
Clifford M. Witcher

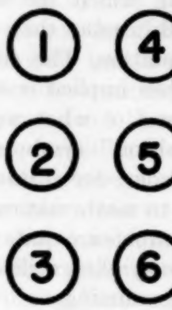


Figure 2

appear on the under side of the paper. Graph paper has been available for a number of years. This paper is ruled off in $\frac{3}{8}$ inch or $\frac{1}{2}$ inch squares whose boundaries are smooth raised lines. Obviously, this is quite crude if one is attempting to perform accurate graphical calculations. Fairly good geometrical instruments, such as straight edges, protractors, compasses, and so on, have been available for some time. The compasses are usually fitted with spur wheels instead of pencil or pen points. A major step forward in the development of equipment for producing raised diagrams of all sorts occurred in 1949, and this will be described a bit later.

The Braille system

The system of raised-dot typography invented by the blind teacher, Louis Braille, in 1829 is of considerable interest, both mathematically and physiologically. Although Braille's development work was entirely empirical, his system is indeed strikingly efficient when analyzed in terms of modern physiology and information theory. In Braille's system there are, as shown in Figure 2, six possible positions for dots in each space for a character, or each "Braille cell," as it is called. By using all possible combinations of dots in these six positions, a total of 2^6, or 64 symbols may be obtained, including the blank space. Braille assigned 25 of them to represent the letters of the alphabet, as shown (Fig. 3). (The letter *w* was not in the French alphabet, and hence had to be added later.) On first considering the alphabet, it might appear that Braille was illogical in failing to assign the simplest symbols (perhaps even single dots in different positions) to the most frequent letters. However, experience has shown that a line of Braille symbols in which too many single dots appear tends to slow down the reader by forcing him to make a series of decisions as to the particular positions in the Braille cell in which the dots occur. Note, however, that the second ten letters (the letters *k–t* inclusive) are formed from the corresponding letters of the first ten by the addition of the same single lower dot on the left. Note also that *u*, *v*, *x*, *y*, and *z* are similarly formed by the addition of still another lower dot on the right. It is of interest to observe that Braille's original choices for the form, size, and spacing of dots have not been altered much in the light of careful modern physiological and psychological studies. The present spacing between Braille dots (0.090–0.095 inch) is just about optimum when considered with relation to the distribution of tactile receptor nerve endings on the finger tip (average separation about 2.0 mm. or 0.79 inch).

Braille perceived that the remainder of the 63 Braille symbols constituted more than enough for conventional punctuation marks, etc. He therefore began assigning symbols to represent frequently encountered words or groups of letters. This idea

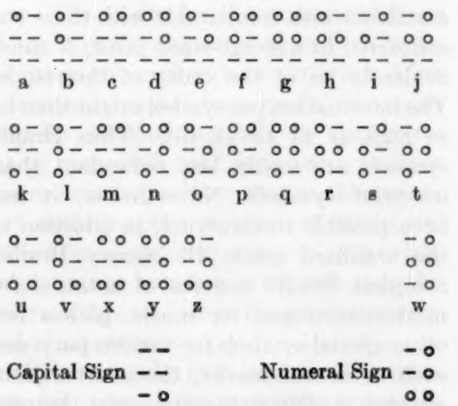


Figure 3

has been carried much further in modern times, frequent use being made of combinations of two symbols, or one special symbol followed by a letter, to represent words or commonly used groups of several letters. The resulting system is now known in English-speaking countries as "grade II Braille." The average reading speed for grade II Braille is 120–150 words per minute. This is, of course, considerably below that for reading by sight, but is far greater than that obtained in any other system of touch reading with one exception. The raised dot system known as "New York Point" approaches Braille in reading speed, but this system has been abandoned for other reasons.

Viewed in the light of modern concepts of information theory and the construction of codes, Braille represents a rather remarkable coding scheme. The amount of information associated with an arbitrarily chosen Braille symbol is exactly 6 "bits," if we include the blank space as a possible symbol. (The unit, or "bit" of information is, by definition, the amount of information represented by one binary choice.) On the other hand, the information that an arbitrarily chosen ink-print symbol represents must be very much greater than 6 bits. It would be equal to $\log_2 N$, where N is the number representing all possible symbols that could be formed by drawing lines of a specified width within a specified area assigned as the maximum for a single symbol. Choosing values of line width and maximum area comparable with those encountered in average-sized print, N must evidently be of the order of thousands. The information per symbol might then be as high as 11 or 12 bits. Thus Braille symbols are vastly less redundant than ink-print symbols. Nevertheless, it has been possible to construct, in addition to the standard grade II literary Braille, complete Braille systems of notation for mathematics and for music, plus a few other special symbols for various purposes, such as accent marks; the constructions are such that no ambiguities exist. As must be immediately apparent, there exist many cases in which the same Braille symbol is used in more than one of these systems of notation. The elimination of ambiguities thus implied is accomplished through the use of what we might call "logical transition" symbols, which instruct the reader, for instance, to shift from literary to mathematical interpretation. These symbols are quite analogous to the instruction symbols utilized in modern computer programming.

Modern teaching methods

Throughout the whole scale of academic attainment, the teaching of mathematics to the blind has undergone vast changes during the past quarter century. In arithmetic, the use of such devices as movable type and the Taylor slate has, for the most part, been superseded by the use of Braille. This has been, mainly, a result of the improved design and increasing availability of Braille writing machines. Modern machines, such as the Perkins Brailler (representing by far the most advanced design), are so easy to operate and so foolproof that they can even be used by children in the first grade of school. Thus many schools are now exclusively using Braille writers for all nonoral arithmetic, algebra, etc.

In many schools the trend in recent years has been toward the abandonment of the conventional formats for written arithmetic in favor of patterns that are more suitable for use in Braille; even by the time the student begins to take up fractions, the duplication of standard ink-print formats becomes quite impractical in Braille. This is because Braille symbols are most conveniently written in a strictly linear sequence. Obviously this necessitates quite radical departures from the conventional formats by the time the student reaches algebra and thereafter.

As might be expected, much more emphasis in the primary grades is placed on the memorization of useful addition and multiplication facts than one usually finds in the primary education of children with normal sight. After all, mental calculation is considerably faster and less laborious

than any sort of written work, and the blind student may therefore save much valuable time if he has thoroughly mastered a number of simple numerical facts.

In recent years some thought has been given to the possibility of utilizing the abacus, or its Japanese counterpart, the soroban, for rapid numerical calculation. In fact, several of these devices have been adapted (mainly by the introduction of a means for applying friction to the counters) for use by the blind. Mr. Edward J. Waterhouse, director of Perkins Institution for the Blind, and himself a mathematician, has expressed the opinion that, if any device for numerical calculation at elementary levels is ever destined to compete again with the Braille writer, it will probably be some variety of adapted abacus or soroban.

A practical form of circular slide rule has recently been developed which is suitable for fairly rapid touch reading. Its development was carried through by Charles G. Ritter and the author about six years ago. It is shown in Figure 4. The disk, 12 inches in diameter, carrying a raised scale near the periphery on each side, is of vinylite composition and is produced by precisely the same pressing process as that used for phonograph records. In fact, a phonograph-record press is used in the production of the disks. The circular scale on one side of the disk is logarithmic; that on the other side is linear. The points of division in both are the same as those of the C and L scales of a 10-inch slide rule. The two pointers on the logarithmic side of the disk are so arranged that they can either be moved independently of each other or locked together to move as a unit. This arrangement permits multiplication and division to be carried out quite simply. On the linear side of the disk is a third pointer very rigidly and accurately aligned with one of the pointers on the logarithmic side, so that, when the pointer on the logarithmic side is set to any given number, the pointer on the linear side will indicate the common logarithm of the number. This permits the computation of

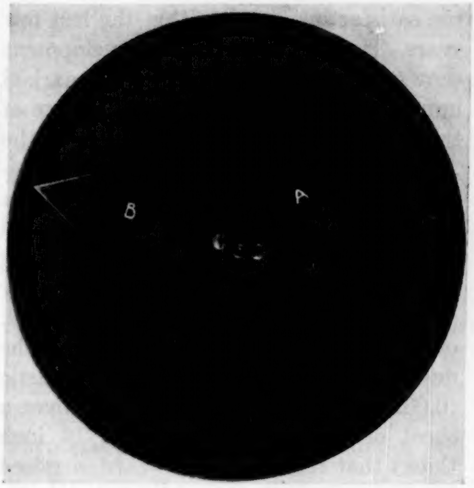


Figure 4

powers and roots with the help of simple mental multiplication or division. This slide rule is now coming into rather extensive use in secondary school mathematics and science courses, as well as by blind college students and scientific workers.

While on the subject of numerical calculation, it may be of interest to note that Marchant and Monroe desk calculators and comptometers have been adapted for touch reading. The adaptation consists in simply fitting over the dials a set of transparent plastic rings carrying raised Braille digits. This scheme, of course, does not interfere with the visual reading of the dials. In some models of the Frieden calculator, no adaptation whatsoever is necessary. The shafts for the dials extend out of the case for a quarter of an inch or so, and their exposed ends are knurled. After a computation has been made, the result may be obtained tactually by simply rotating each shaft back to the zero position (there being a positive stop at this point) and counting the number of clicks required for each shaft rotation. The only disadvantage here lies in the fact that the dial readings are lost in the process of tactual observation.

As has already been stated, the situation with regard to geometric constructions, graphs, and the like, has not been

too satisfactory, even within the last few years. However, a recent development shows promise of producing a marked improvement in this area. In 1947 we of the technical research department of the American Foundation for the Blind decided to make one more attempt to find a means of drawing that would yield raised lines on the top side of the drawing medium. A little more than a year later an effect was discovered by H. P. Sewell, of New York City, which achieves the desired result. If a thin sheet of plastic (0.002–0.004 inch thick) is placed over a sheet of gum rubber (about 1/16 inch thick) that rests on a flat board, a raised line appears on the top side of the plastic when drawing is done with a rounded point, such as that of a ball-point pen. The explanation, once it was discovered, turns out to be quite simple. Most plastics are subject to cold flow when forces are applied to them. Thus the force applied by the point of the drawing instrument stretches the plastic in the direction of motion of the pen, but leaves the remainder of the sheet undistorted. The result of the stretching is to produce a series of gathers in the direction perpendicular to the plane of the sheet. Because of the presence of the rubber below the plastic, the distortion will be mainly in an upward direction, thus producing a raised line on the top side of the plastic. Since the line is composed of a series of gathers, it produces a tactile impression consisting of a series of closely spaced dots. Both cellulose acetate and a variety of moisture-proof cellophane have proved to be quite satisfactory drawing media. By varying the pressure of the pen on the plastic, lines of variable height and width may be obtained.

The new system has proved itself to be quite useful for the production of all sorts of diagrams, graphs, and so on, either by the blind themselves or by seeing friends. It lends itself nicely to applications as diverse as the teaching of handwriting and the production of electronic circuit schematics.

Of all the recent developments in the field of mathematics for the blind, perhaps the most significant has been that of the new system of Braille mathematical notation. This advance is due to Mr. Abraham Nemeth, of New York City, a blind graduate student in mathematics at Columbia University. Although the system was only completed about two years ago, it has met with such enthusiastic reception throughout the country that mathematics texts incorporating it have already been produced, and last year it was introduced into twelve schools for the blind.

The system of notation previously in use in this country and in England was devised before the turn of the century by William Taylor, mentioned earlier as the inventor of the Taylor slate. Although this system contained a few good features, it was, for the most part, quite clumsy. Among other things, it necessitated far too extensive use of symbols of grouping, such as parentheses and square brackets. Besides, the older notation scheme contained absolutely no provisions for handling any sort of mathematics more complex than elementary algebra and trigonometry.

Nemeth's system retains all of the desirable features of the Taylor notation. In fact, at the levels of elementary arithmetic and algebra, it does not differ from Taylor's notation radically except through the incorporation of a few simplifying and space-saving devices. However, even at the level of intermediate algebra, the power and simplicity of the new system become quite striking.

One very intriguing feature of the new notation is its way of dealing with exponents and subscripts. As has already been said, the only practical way of writing Braille is in the form of a linear sequence of symbols. Nemeth therefore conceived the idea of so-called "level indicator" symbols, which are again quite analogous to some of the coding devices utilized in modern computer programming. One such symbol instructs the reader to shift the level of succeeding characters up by one step. Thus x^n would be expressed by

writing x followed by this level shifting symbol followed by n, followed finally by the symbol that signifies a return to the main line. Other symbols exist for shifting the levels of characters up two steps or down either one or two steps. Excessive use of symbols of grouping is eliminated by such devices as the use of special initiation and termination symbols for fractions and for radicals. Thus, when the reader reaches the symbol that initiates a fraction, he knows that all subsequent characters will be those of the numerator until he reaches the symbol that indicates a shift to the denominator. Thereafter, all characters must be elements of the denominator until he reaches the fraction termination symbol. To write a simple fraction in the older notation required, in general, a pair of symbols for square brackets plus two pairs of symbols for parentheses plus the symbol indicating transition from numerator to denominator.

There is very little mathematics that cannot be handled by the new system. Careful analysis of it has indicated that it is quite adequate for such things as modern algebra, vector analysis, theory of functions. And so we see how fortunate is the blind mathematics student of today when compared with the students of the age of Nicholas Saunderson!

Have you read?

ORLEANS, JACOB S. and SPERLING, JULIA L. "The Arithmetic Knowledge of Graduate Students," *Journal of Educational Research*, November 1954, pp. 177–186.

As mathematics teachers you are all aware of the fact that difficulty in algebra may in reality be difficulty in arithmetic. The authors of this article felt that a similar situation exists for students taking statistics applied to education and arithmetic. To verify this, a study was made of the written arithmetic calculations which 72 students of statistics made while taking an examination in the course. The instructor had stressed mental arithmetic, the use of tables and various simplified procedures for arithmetical solutions. It will surprise you to know that problems like 12 divided by 2, 6 divided by 5, 20 divided by 4, etc., were written out in short division form. Long division was used for 183 divided by 4, 7 divided by 2, etc.; 950 was divided by 4 by long division, as well as 1950, 2950 and 3950. The problem

$$\frac{6}{\underset{3}{\cancel{9}}}\times\overset{2}{\cancel{6}}=\frac{12}{3}=4$$

was written out rather than calculated mentally. Many other similar problems were written out completely. Addition of 100 and 20 was written out formally, as were subtraction and multiplication of such simple problems. This next item is difficult to believe, but with a table of squares and square roots provided for their use, 57 of the 76 people worked these problems out rather than using the table, and in several instances they got an incorrect result. You should read this for the many illustrations given and then, with me, you will wonder if we are teaching a functional arithmetic.

BEHARI, RAM. "Presidential Address," *The Mathematics Student*, January 1954, pp. 11–26.

It is my opinion that this address, presented before the Indian Mathematical Society at the University of Delhi in 1953, will do for India what E. H. Moore's presidential address, "On the Foundation of Mathematics," given over 50 years ago before the American Mathematical Society, did for the United States. He believes mathematics leads to a liberation of the mind; that it provides men with new ways of thinking; that it has provided new spaces and consequently leads us into a different culture. He believes that a nation without modern mathematics is like a well-dressed man without character. To neglect mathematics is to commit national suicide.

You will also note that problems on the other side of the world are very much like ours. The curriculum is cluttered with nonessentials, trick problems, Horner's method, and others, to the exclusion of modern, efficient mathematics. The pay scale for mathematics teachers is too low. The teachers are required to do computational arithmetic for the school, such as finding percent of attendance, test averages, and the like. You will say amen to his suggestions: a raise in salary, more time for study, and standardization of symbols. You'll profit from and enjoy this address.—PHILIP PEAK, *Indiana University, Bloomington, Indiana*

The calculation of logarithms in the high school

CARL N. SHUSTER, *State Teachers College, Trenton, New Jersey.*

An intuitive approach to logarithms which high school teachers may find helpful.

WITHOUT DOUBT the most widely accepted assumption in modern mathematics education is that the *why* must be taught before, with, or at least very shortly after the *how*. At one time, very little attention was given to the *why*. The *how* was taught as thoroughly as possible and then a large amount of drill was given.

A	0	1	2	3	4	5	6	7	8	9	10
B	1	2	4	8	16	32	64	128	256	512	1024
C	2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}

We are quite sure today, however, that if the pupil thoroughly understands the mathematical principles behind each process, far less drill will be needed. The pupil will enjoy his work and there will be far more transfer.

To make logarithms and their principles meaningful, it is a good plan to start with logarithms to the base 2.

If the three rows given above extended to 2^{20} or 2^{30} the results could be quite spectacular.

Using A, B, and C it is quite easy to develop the rules for using logarithms in multiplication, division, roots, and powers. The pupil can see the relation of the logarithm (A), the antilogarithm (B), and the exponent (C). He can see that a logarithm is an exponent showing the power to which the base must be raised to produce the antilogarithm.

Another very important fact that the pupil should notice is that if we have two pairs (log and antilog), it is possible to find the pair between them. That is, if we have (3, 8) [Note: 3 log, 8 antilog] and (5, 32) we may find the pair (4, 16). The logarithm 4 will be the arithmetic mean of the logs 3 and 5. That is $4=(3+5)/2$. The antilog 16 will be the geometric mean of the antilogs 8 and 32. That is $16=\sqrt{8\times 32}$.

This should be tested with several other sets. This fifth fact will be used later. After a day or so spent on logarithms to the base 2, the pupil should be shown that 3, 4, 5, or any other base could be used but that since our number system is based on ten the base ten is by far the most efficient base to use. The pupil should then work out a set similar to A, B, and C using the base 10.

A	0	1	2	3	4	5
B	1	10	100	1000	10000	100000
C	10^0	10^1	10^2	10^3	10^4	10^5

At first the pupil may feel that the above set is not as efficient as the first set since the numbers are so far apart. At this time he should be introduced to a table of logarithms (or mantissas) and told that it is possible to find with the table the logarithm of any number. However, all the rules developed experimentally with the logarithms to the base 2 still hold.

Most pupils will be rather curious as to how the table of mantissas was constructed. Up to this time in their mathematics it has been possible to show them how each table they have used could be constructed. It will not satisfy the pupils to be told that they must learn calculus or use one of the giant mechanical computers to work out a logarithm. Professor Ransom[1] has given a very interesting method for finding the approximate logarithms of numbers 3, 5, 7, 11, . . . 47. Another method by which the actual logarithms of certain numbers correct to any desired accuracy can be obtained, depends on the fifth fact developed with the logarithms to the base 2. Using the series *A*, *B*, and *C* with logarithms to the base 10 it can be shown that from the pairs (1, 10) and (3, 1000), the pair (2, 100) may be obtained:

$$\frac{1+3}{2}=2 \quad \text{and} \quad \sqrt{10\times 1000}=100$$

The pupils will be quite willing to postulate that the rule will hold between any two sets. That is, if we take (1, 10) and (2, 100) then

$$\frac{1.000000+2.000000}{2}=1.500000$$

and

$$\sqrt{10.0000\times 100.000}=31.6228$$

(Note from a large table: log 31.6228 =1.500000.) We may now take 1.50000, 31.6228 and 2.000000, 100.000 and find the pair 1.75000, 56.2342. By constantly working for the pair midway between the pairs found, the following can be obtained:

NUMBER	LOG COMPUTED	LOG FROM TABLE
32.6228	1.500000	1.500000
56.2342	1.750000	1.749998
74.9898	1.875000	1.874999
86.5966	1.937500	1.937501
93.0573	1.968750	1.968750

If the slide rule-division method is used for square root[2] the above pairs may be computed in a very few minutes. These values should be compared with the values in a large table, or checked by using the principles the pupils have developed. Perhaps it might be well to show them that using the five pairs obtained, they could find a large number of other pairs. The product of

$$86.5966\times 93.0573=8058.45.$$

Also

$$1.93750+1.96875=3.90625 \text{ (log)}.$$

[From a large table log

$$8058.446=3.9062510.]$$

If no other method for computing logarithms was available, it would be possible to build an entire table by the above methods. The pupils should now be told that where an entire table is to be computed, a number of other methods that they will learn in college are far more efficient.

The satisfaction that high school pupils get from knowing that they can compute the value of certain logarithms to any desired degree of accuracy is well worth the time spent. The mystery of logarithms will have vanished and the increased power due to comprehension will be a fitting reward for both pupil and teacher.

[1] William R. Ransom, "Elementary Calculation of Logarithms," The Mathematics Teacher, Vol. XLVII, No. 2 (February 1954), p. 115.

[2] C. N. Shuster, "Approximate Square Roots," The Mathematics Teacher, Vol. XLV, No. 1 (January 1952), p. 17.

Student discovery of algebraic principles as a means of developing ability to generalize[1]

OSCAR SCHAAF, *Eugene High School, Eugene, Oregon.*

The ability to generalize (and to specialize) is an important objective in any course in mathematics. Can it be successfully taught in the secondary school?

ONE PROBLEM confronting secondary schools is the determination of a suitable mathematics program for students in the ninth grade. This problem becomes more acute when all ninth-grade students are enrolled in the same mathematics class, and in this case it is important that both content and teaching methods provide a wealth of educational value for *all* students. An attempt to conduct a course for such a class was made at the University School, Ohio State University, Columbus, Ohio, where all thirty to thirty-five ninth-grade students were enrolled in the same mathematics class. It was felt that this course should not only have a theme which embraces the spirit of mathematics but also should be consistent with the general purposes of the school. The theme chosen, improvement in ability to generalize, seemed to meet this criterion, and its selection determined in a general way the nature of class teaching procedures to be used. Improvement in generalizing ability is probably most certain if students actually practice generalizing. The practice can be provided if students discover for themselves as much as possible the mathematical principles to be learned. However, students are not expected to rediscover independently the mathematics which took centuries to be discovered. Instead, they are given the type of guidance and mathematical experiences which will enable them to become conscious of the emergence of "new" mathematical concepts and principles from their past and present experiences. In addition to providing this guidance and helping to select these experiences, the teacher's role is to remove obstacles that prevent students from generalizing, to encourage students in continuing their "research" by supplying timely hints as needed, and to lead students in seeing the similarity between discovery procedures in mathematics and procedures as used in nonmathematical situations.

Care must be exercised in selecting the content of such a course since it is likely to be the terminating one for many students but for others a preparation for more advanced mathematics. Guiding principles used in the selection of content for the experimental course were as follows:

1. Students should study mathematical content which can be used to illustrate the various generalizing procedures.

2. Students should be given an overview of mathematics and become aware of the interrelatedness of its many branches.

3. Students should study topics they can make use of now and most likely will use in the future.

4. Students should study topics which

[1] A doctoral study supervised by Professor Harold P. Fawcett, Chairman of the Department of Education, Ohio State University.

review and extend their understandings of arithmetic.

The actual content studied by the experimental class can be summarized under the following headings:

1. Number and operation
2. Graphs and formulas
3. Equations and problem solving
4. Proportion and indirect measurement
5. Statistics

Four tasks were assumed to be of the most significance in carrying out this study: an analysis of the different processes of generalizing and determining the characteristics of a superior generalizer; a formulation of lesson sheets and other procedures which can be used in aiding students to develop their ability to generalize in both mathematical and nonmathematical situations; an evaluation, in terms of the characteristics of a superior generalizer, of any changes made by students in the experimental class; and an evaluation of the mathematical achievement of the experimental class.

Methods of generalizing were classified under the major divisions of empirical and rational procedures. Under empirical procedures sub-classes were simple enumeration, analogy, continuity of form, and statistical procedures; under rational procedures the sub-classes were deduction, variation, formal analogy, and inverse deduction. The analysis of these procedures revealed that the superior generalizer should

1. be able to detect likenesses and differences between situations and then to group together those situations which possess common properties;
2. be able to generalize correctly and with certainty where certainty is justified, but refrain from generalizing when it is unwarranted;
3. be able to determine trends from available data and from them make reasonable extrapolations and interpolations;
4. be cautious in accepting generalizations, but willing to consider any generalization as a hypothesis;
5. distinguish generalizations from observable facts;
6. use generalizing methods independently to further his understanding of the world about him;
7. make use of generalizations which he senses through nonverbal means;
8. be able to verbalize generalizations and methods used in discovering generalizations;
9. continually search for examples which are exceptions to formulated generalizations and then reformulate them to exclude the exceptions;
10. investigate implications of generalizations under consideration;
11. search for explanations for generalizations discovered empirically;
12. empirically test or search for an increasing number of applications of generalizations that have been suggested by rational procedures;
13. search for and recognize relevant factors responsible for happenings or for behaviors of objects in certain situations;
14. study accepted generalizations from as many viewpoints as possible; and
15. use statistical measures in making generalizations.

Many different approaches which lead students to the discovery of mathematical principles were used on the lesson sheets designed for the experimental class. Examples of two different approaches will be given here. The first, signed numbers 9, helps students in the discovery of generalizations concerning order of operations. Before being given this lesson, students had studied both addition and subtraction of signed numbers and they knew the function served by brackets. Of course, the principle to be discovered here is that if the addition sign precedes the brackets their elimination would have no effect on the final answer, but if the subtraction sign precedes the brackets, then their elimination would result in a different answer.

SIGNED NUMBERS 9

Perform the indicated operations and give the result.

1. $+8+(+5)-(+6)=$
2. $+8+[(+5)-(+6)]=$
3. $+6+(-8)-(-5)=$
4. $+6+[(-8)-(-5)]=$
5. $+12-(-8)+(+9)=$
6. $+12-[(-8)+(+9)]=$
7. $-3-(-13)-(-21)=$
8. $-3-[(-13)-(-21)]=$
9. $+7+(-15)+(-22)=$
10. $+7+[(-15)+(-22)]=$
11. $+16-(-26)+(+18)=$
12. $+16-[(-26)+(+18)]=$
13. As you probably have noticed the problems above are given in pairs. The only difference between each example in the pair is in their order of operations. Does the order of operations make a difference always? Some of the time? Never?

Write a statement of your conclusions concerning order of operations.

The second illustration attempts to lead students in the discovery of the rule for division of signed numbers and is given after students know the rule for the other operations with signed numbers.

SIGNED NUMBERS 14

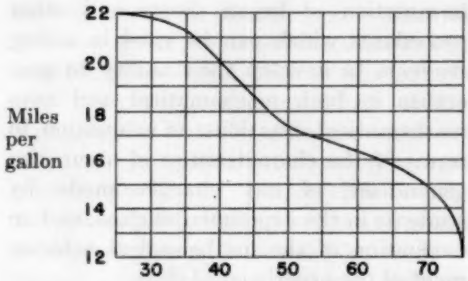
1. $(+7)(\quad)=+35$
2. $(\quad)(-8)=-32$
3. $(-5)(\quad)=+30$
4. $(+4)(\quad)=-56$
5. $(\quad)(-3)=+93$
6. $(+6)(\quad)=-5.4$
7. $(\quad)(-15)=240$
8. $(.2)(\quad)=10$
9. $(\quad)(24)=-256$
10. $(\quad)(-.2)=3.2$
11. $(-96)(\quad)=+12$
12. $(56)(\quad)=-49$
13. How did you find the missing numbers in the above exercises?

So far, you have discovered the rules for addition, subtraction, and multiplication of signed numbers. The exercises below are division problems. Write down what you think their answers are.

14. $\dfrac{39}{-3}=$
15. $\dfrac{-84}{7}=$
16. $(+56)\div(-7)=$
17. $(-64)\div(-16)=$
18. $\dfrac{-15}{-3}=$
19. $(-27)\div(+3)=$
20. $\dfrac{0}{+6}=$
21. $\dfrac{+8}{0}=$
22. $(-12)\div(0)=$
23. $(0)\div(-5)=$
24. What is the rule for division of signed numbers?
25. Write problems 1 through 6 as division problems.

In addition to those leading to the discovery of mathematical principles, other lesson sheets were designed in which students were to generalize in situations largely nonmathematical in nature. Below is one such lesson.

INTERPRETING GRAPHS



Miles per hour—at constant speeds
Buick Special with Dynaflow

Mark it *1*—if the *facts given* make the statement true.
Mark it *2*—if the *facts given* make it probably true.
Mark it *3*—if you cannot tell from the *facts given.*
Mark it *4*—if the *facts given* make it probably false.
Mark it *5*—if the *facts given* make the statement false.

____ 1. The faster the Buick Special (with Dynaflow) travels, the more miles per gallon of gasoline.
____ 2. It is more economical to drive a Buick than a Cadillac.
____ 3. The number of miles per gallon for a Buick Special (with Dynaflow) averaging 80 m.p.h. is 11.9 miles per gallon.
____ 4. Buick is the best buy on the market.
____ 5. The number of miles per gallon of gasoline for 30 m.p.h. is twice as much as it is for 60 m.p.h.
____ 6. The faster a car travels, the less the number of miles per gallon of gasoline.
____ 7. The number of miles per gallon at 35 m.p.h. is approximately 21 miles.
____ 8. At twenty miles per hour the Buick Special (with Dynaflow) uses less gasoline per mile than at 25 m.p.h.

——— 9. The Buick Special (with Dynaflow) uses more and more gasoline per mile as the speeds increase from 25 m.p.h. to 75 m.p.h.

———10. Buicks without the Dynaflow get more mileage per gallon than Buicks with Dynaflow.

Evidence from observers' reports, student notebooks, the teacher's notes concerning each class meeting, and responses on student reaction sheets indicated that the teaching procedures and the lesson sheets designed for the experimental course did work toward the acquisition of the abilities and behaviors listed earlier. To obtain a more objective measure of the students' ability to generalize, a generalization test, which was essentially nonmathematical in nature, was designed and given at the beginning and at the end of the school year to the experimental group at the University School and also to a status group made up of algebra classes in the Columbus Public Schools. Test results suggested the conclusions that members of the experimental class

1. made significantly greater improvement in their ability to draw conclusions which are justifiable extrapolations and interpolations of accepted data;
2. made significantly more improvement in their ability to recognize conclusions that were not justifiable extrapolations and interpolations of accepted data;
3. became noticeably more cautious when generalizing from data both surrounded by and relatively independent of an emotive context; and
4. made significantly greater improvement in interpreting graphical and tabular data.

In order to determine the effect of emphasizing improvement of generalizing abilities in both mathematical and nonmathematical situations on the students' algebraic achievement, an aptitude test was given to the experimental class at the beginning of the year and two algebraic achievement tests at the end of the year. Achievement as measured by results on the Ohio Every Pupil Test was accurately predicted by results on the Iowa Algebra Aptitude Test. Their achievement as measured by the Lankton First Year Algebra Test was significantly greater than was predicted by the results on the Iowa Algebra Aptitude Test. This achievement was accomplished even though these students had much less school time for the study of algebra than do students in the algebra classes of the public schools.

It seems reasonable to suggest from the results of this study that students in the experimental class, by and large, made significant improvement in their ability to generalize which, in part, was due to the teaching procedures employed in the mathematics class. Also, the experimental students' mastery of algebraic concepts was equal if not better than if they had been taught algebra by conventional teaching procedures.

Algebra or arithmetic?

Three authors agree to share the royalties on a book as follows: At first Rider gets $\frac{1}{4}$, Hoelf gets $\frac{1}{4}$, and Wright gets $\frac{1}{2}$ of the royalties. This goes on until Rider has received \$1000. From then on, Hoelf gets $\frac{1}{3}$ and Wright gets $\frac{2}{3}$.

After several payments have been made, the publisher has \$1800 to divide up among the authors. Giving Rider $\frac{1}{4}$ of this would run his total above \$1000. So the publisher pays Rider enough to bring his total to \$1000.

The publisher considers 2 plans for paying Hoelf and Wright their fair shares of the \$1800:

(1) Give Hoelf what Rider got. Give Wright twice what Rider got. Then divide what is left so Hoelf gets $\frac{1}{3}$ and Wright gets $\frac{2}{3}$.

(2) Give Hoelf $\frac{1}{3}$ of what is left after paying Rider. Give Wright $\frac{2}{3}$ of what is left after paying Rider.

Compare these plans. Does one of the plans favor Wright? Or are they really equivalent? Use algebra if you like. But don't stop short of saying "of course it 's obvious, because . . .

When Hoelf gets $\frac{1}{4}$ and Wright gets $\frac{1}{2}$, they share on a 1 to 2 basis. So, in either plan, Hoelf gets $\frac{1}{3}$ and Wright gets $\frac{2}{3}$ after Rider is paid.

Words! Words! Words!

This is a plea for sympathetic understanding and serious study of our problems of terminology. It constitutes one reader's reaction to the comments "Words! Words! Words!" on pages 81 and 89 of the February issue.

As teachers of mathematics we must make a constant effort to teach mathematics as a living subject, an area of knowledge in which new concepts are constantly arising. There are now many efforts to keep our courses from becoming studies of a static discipline inherited from the Middle Ages. These efforts should include our terminology as well as topics in our courses.

Consider, for example, the concept of a *function* as a set of ordered pairs. This concept has some critics even among research mathematicians. It also enjoys considerable popularity. Some secondary school teachers could make effective use of the ordered pair terminology; others would find it meaningless to their students. However, nearly all teachers of secondary school mathematics are aware of the importance of the function concept. They use it frequently in their classrooms. Even though the recent developments at the research level may not be directly applicable to most high school classes, these developments have the following important implication for all mathematics teachers. The word *function* is now used as synonymous with *single valued function*.

The concept of a *variable* is also in the process of being clarified. In this case no obvious solution to the related problems of teachers of high school algebra has yet appeared. From the point of view of modern algebra variables do not exist since the traditional theoretical concept of a variable has been "shot full of holes." The purpose of this "letter to the editor" is to make a plea for sympathetic understanding of sincere efforts to use recent mathematical and educational theories to find a revision of the concept of a "variable" suitable for secondary schools. This is not a defense of any presently proposed solution. Further study and a great deal more classroom experience is needed before a solution can be expected.—Bruce E. Meserve, Montclair, New Jersey

Editor's note: Bruce Meserve has a good point. The readers of The Mathematics Teacher would like to know what the "holes" are in the traditional concept of a variable.

Have you read?

Maclean, John. "Why Teach Mathematics?" *The Mathematics Gazette*, May 1954, pp. 96–110.

This article is of a great deal of interest to the mathematics teachers in the United States because it points out the similarity of our problems and those of the English modern high school.

Mr. Maclean offers some excellent suggestions on the direction our teaching should take. For example, he asks why we take pages to give verbal descriptions when simple mathematical statements could be used if people were properly prepared; why use complicated arithmetical computation when simple algebraic computation is available; why learn many special techniques when one basic principle enables one to do them all? He offers many other similar suggestions. You will be interested to note his comments on unification of mathematics; in fact, he says the test of teaching is, "have we imparted a feeling of unity in the pupil's outlook, a sense of wholeness?" He is concerned about the postponement of certain mathematics because we are only capable of doing a part of it; that we do not sufficiently emphasize the function aspect. You will be pleased with his discussion of graphing and how it can be used to present clearly the concept of variation, and also bring forth an evaluation. His interest in applications to other fields and how these fields will profit is of value to all of us. This article will do much toward making us take another look at our present mathematics curriculum. —Philip Peak, *Indiana University, Bloomington, Indiana*

Edited by Emil J. Berger, Monroe High School, St. Paul, Minnesota

Casting geometric solids in plaster-of-Paris

by Wallace L. Hainlin, Horace Mann Junior High School, Miami, Florida

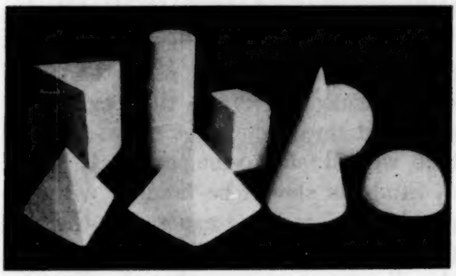


Figure 1

Teachers of junior and senior high school mathematics occasionally experience difficulty in obtaining suitable models of common geometric solids for classroom use. Commercially made sets are ordinarily quite expensive, and many times it is impractical to have models made in the school shops. One fairly satisfactory solution to this problem is to cast such models in plaster-of-Paris by making use of paper models. A teacher or student should be able to cast a very suitable set of models in one afternoon (Fig. 1).

The first step in the manufacture of a set of molds is to collect suitable molds or to fashion them from paper or cardboard and painters' masking tape. Ordinary household items can often be made to serve very nicely. A pasteboard mailing tube may be used as a mold for a right circular cylinder, a conical paper cup for a cone, an ordinary paper cup for a frustum of a cone, and a small pasteboard box for a rectangular prism. As already indicated, some molds, such as those needed for a pyramid, a tetrahedron, or other multi-sided solid, may have to be made from cardboard or paper and masking tape. A mold for a hemisphere may be made by wrapping heavy aluminum foil around a small rubber ball and then carefully cutting the foil with a razor blade to remove the ball. A sphere may be made by "plastering" two hemispheres together with plaster-of-Paris.

After the molds have been collected, they should be supported in an upright position and filled with a mixture consisting of two parts plaster-of-Paris and one part water. Allow about one hour for the plaster to set; then peel off the molds and sandpaper the surfaces to remove imperfections. If the models are to be painted, at least one full week should be allowed to elapse following the casting.

Plaster-of-Paris models, although not as durable as wooden ones, will last for many years if they are properly handled. One way of extending their life is to cast a short wooden handle in each. Another way is to reinforce such parts as the vertices before pouring the plaster-of-Paris.

Anyone who has a learning aid which he would like to share with fellow teachers is invited to send this department a description and drawing for publication. If that seems too time-consuming, simply pack up the device and mail it. We will be glad to originate the necessary drawings and write an appropriate description. All devices submitted will be returned as soon as possible. Send all communications to Emil J. Berger, Monroe High School, St. Paul, Minnesota.

Using the tower of Hanoi to present the principle of mathematical induction

by William Koenen, Sault Ste. Marie High School, Sault Ste. Marie, Michigan

Recently it occurred to this writer that the Tower of Hanoi and the rules that make up the problem associated with it might possibly be used to introduce the principle of mathematical induction in an easily understood presentation at an elementary level of instruction.

For the convenience of the reader, we have included a brief description of the device and a statement of the rules that make up the problem. The device consists essentially of three identical, vertical dowel pegs mounted on a rectangular wooden base (Fig. 3). An arbitrary number of discs varying in diameter from, say, two to four inches, and with holes drilled at their centers should be slipped onto one of the pegs. To make the problem seem formidable at least six such discs should be employed. At the outset, the discs should all be on one peg and arranged in order of size with the largest on the bottom.

The object of the problem associated with the device is to locate all the discs on one of the remaining two pegs in accordance with the following rules:

a. Only one disc may be moved at a time.
b. In moving a disc, it must be removed from one peg and placed on another, but never over a smaller disc.

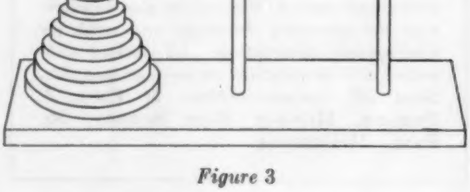


Figure 3

In a demonstration before students, appropriate gestures would allow the use of such language as, "from this peg to this peg," but since we cannot utilize gestures here, we will refer to the pegs as A, B, and C. Students should be familiar with the problem before being exposed to the proposed demonstration. To continue with this discussion, assume that the discs are initially on peg A.

To move any disc from its original position on A to peg C, all the smaller ones above it must first be located on peg B. (In a demonstration the reason for this can be enlarged upon by moving the discs to the appropriate positions.)

Suppose it is agreed that one particular disc can be moved to C, and that by repeating the same series of moves that brought the smaller discs to B, these smaller ones can be moved on top of the first mentioned disc on peg C.

That is, if there exists a series of moves that will put the k smallest discs on peg B (from A), then the $(k+1)$st disc can be moved to C. And using the same series that moved the k smallest to B from A, these k can be moved to C (on top of the $(k+1)$st disc) from B. After this has been accomplished, there are $(k+1)$ discs on peg C. Thus, we have shown that *if* a series of moves exists that will put the first k discs on peg B, then there exists a series of moves that will put the first $(k+1)$ discs on peg C.

We can verify the obvious fact that there exists a series (of one move) that

will put the first *one* disc on peg B. Therefore, by the above argument there exists a series of moves that will put the first two ($one+1$) discs on C. Since the labels B and C of the pegs are of no significance, this argument can be re-applied indefinitely and we can establish that any number of discs can be moved from their original position on peg A to peg B (or C, depending on how they are labeled).

A model for visualizing the Pythagorean theorem

by Clarence Olander, St. Louis Park High School, St. Louis Park, Minnesota

The Pythagorean relationship is one of the most widely used of all mathematical formulas. Accordingly, junior high school classes in algebra and general mathematics usually include some work involving simple applications of this important relation.

In this note we suggest a model as one means of familiarizing junior high school students with the concept that $a^2+b^2=c^2$. where, by convention, the hypotenuse is represented by the letter c and the two legs by the letters a and b.

To construct the model, select a piece of masonite $\frac{1}{8}'' \times 8'' \times 8''$, draw a small right triangle on its surface and cut out the squares on the three sides with a keyhole saw (Fig. 2). Mount the piece of masonite with the cut-out squares on a piece of board $\frac{3}{4}''$ thick. Nail the right triangle resulting from cutting out the squares on the board in the position indicated. Note that two of the vertices of the right triangle are slightly rounded. This rounding is necessary because small BB's must be able to pass from each little square to the large one. Next, paint each square on the wooden base in a bright color. A different color should be used for each square (e.g.,

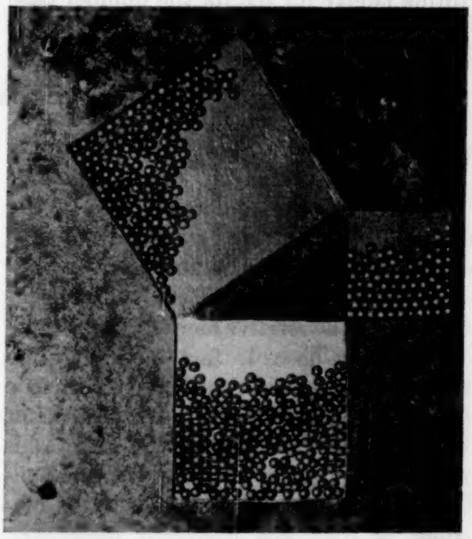


Figure 2

red, green, and yellow). Finally, completely fill the square on the hypotenuse with BB's and cover the device with a thin sheet of plastic $\frac{1}{8}''$ thick. Fasten the plastic at the corners with screws as shown in Figure 2.

In order to attain maximum learning, each student should have an opportunity to manipulate the model by himself.

Edited by Phillip S. Jones, University of Michigan, Ann Arbor, Michigan

The twelve base[1]

by Ralph H. Beard, President, Duodecimal Society of America

For almost as many years as we have used our present system of number notation, it has been known that ten was not necessarily ten, but could represent two, or twelve, or any number that was used as the base of the number system. Ever since then, there has been continuing consideration of the possible advantages in substituting twelve for ten for the number base for general use.

Today, we realize that any efficient number system must include an ordered arrangement of weights and measures, conforming to the same base. If twelve were the base, many of our old familiar measures fall into ordered places in the system, such as the yard, the foot, the inch, the pint, and the pound.

There has been a great amount of argument over the relative merits of the twelve and the ten-base. In fact the contention for the adoption of the French decimetric system is part of the same argument. But no amount of argument will solve this problem. Only proved and preferred application to practical use will bring a solution.

Mathematics teachers will readily comprehend the subtle penetration of the number-concept into all knowledge. Valuations of force, time, angle, and dispersion are as much involved as mass, number, and size. Recently the eruption of electronic computing devices has focused attention on the application of the binary base to problems as diverse as logic and machine control. This past year, the development of the three-stage relay has introduced the use of the trinary base to this field. In addition to black and white, we can now consider gray.

The binary base is too limited for satisfactory general use. Conversions into and out of the ten-base, for binary machine operations, are wasteful of expensive electronic equipment. The three-stage relay makes possible the use of the twelve-base for computer application without similar waste.

These factors have been stated as an introduction to what is the essence of the problem. We condition our children into ordering their thinking around a ten-grid. We could be very wrong. Factorability and flexibility may be more important than we have realized. Number is a sort of solvent for knowledge, and the ten-base may be lacking in essential fluidity.

In the long-range handling of this question, and in training our youth to be familiar with the use of various number bases, mathematics teachers will find a valuable reference aid in the duodecimal bibliography, published in the *Duodecimal Bulletin* for October, 1952. The Duodecimal Society[2] will supply copies of this issue on request, without charge to mathe-

[1] This note grew out of a correspondence with Mr. Beard stimulated by the article, "The Binary System," THE MATHEMATICS TEACHER, XLVI (Dec. 1953), pp. 575–577.

[2] The Duodecimal Society, 20 Carlton Place, Staten Island 4, N. Y. A bibliography by William L. Schaaf on "Scales of Notation" appeared in THE MATHEMATICS TEACHER, XLVII (Oct. 1954), pp. 415–417.

matics teachers, and similar distribution is made of the introductory duodecimal pamphlet, *An Excursion in Numbers*, by F. Emerson Andrews.

Included in the bibliography is a chronological list of the duodecimal writings prior to 1800. This chronology is reprinted here to emphasize the serious consideration that has been devoted to this problem by many good minds, over many years.

DUODECIMAL CHRONOLOGY
PRIOR TO 1800

1585 Simon Stevin, *L'Arithmétique*
1610 Thomas Harriot, *Binary Numeration*
1640 Pierre de Fermat, Letter to Frenicle
1665 Blaise Pascal, *De Numeriis Multiplicibus*
1670 Joannis Caramuel, *Vetus et Nova*
1676 G. W. Leibniz, Mss. in Bibliothek Hannover
1687 Joshua Jordaine, *Duodecimal Arithmetic*
1719 Johann F. Weidler, *Dissertatio*
1731 Edw. Hatton, *Entire System of Arithmetic*
1740 Christoph F. Vellnagel, *Numerandi Methodi*
1747 Ioannem Berckenkamp, *Leges Numerandi*
1760 Buffon, *Essai d'Arithmétique Morale*
1764 Bezout, *Cours de Mathématique*
1784 Encyclopédie, Méthodique, *Echelles Arithmétiques*
1795 Lagrange, *Leçons elementales*
1796 Laplace, *Exposition du Systeme du Monde*
1799 J. E. Montucla, *Histoire des Mathématiques*

America's first mathematician

by Phillip S. Jones

The tale of Nathaniel Bowditch, sea captain, scholar, businessman, astronomer, and mathematician not only personifies much of New England's early days but also has in it such elements of romance that it has been the source of one novel[1] and several biographies,[2] all of which are interesting reading.

It has been said that "mathematical research in America did not begin before the early part of the nineteenth century, when Nathaniel Bowditch (1773–1838) and Robert Adrian (1775–1843) made a definite though small beginning."[3] These two men had in common an interest in astronomy and gravitation which were appropriate to a period of commercial expansion and exploration and of intellectual interests related to Newtonian mechanics and mathematics. However, whereas Adrian was born in Ireland and in this country was connected with the University of Pennsylvania, Rutgers, and Columbia,[4] Nathaniel Bowditch was born in Salem and never taught in a university although he refused professorships at Harvard, the University of Virginia, and the United States Military Academy.

Nathaniel Bowditch, whose father was

[1] Alfred B. Stanford, *Navigator; The Story of Nathaniel Bowditch* (New York: W. Morrow and Co., 1927).

[2] R. E. Berry, *Yankee Stargazer* (New York: McGraw-Hill Book Co., Inc., 1941); Dirk J. Struik, *Yankee Science in the Making* (Boston: Little Brown and Co., 1948), pp. 66–78, 174–176; R. C. Archibald, "The Scientific Achievements of Nathaniel Bowditch" in *A Cataologue of a Special Exhibition of Manuscripts, Books, Portraits, and Personal Relics of Nathaniel Bowditch* (Salem, Mass.: Peabody Museum, 1937).

[3] R. C. Archibald, *A Semicentennial History of the American Mathematical Society 1888–1938* (New York: 1938), p. 1.

[4] D. E. Smith and Jekuthiel Ginsburg, *A History of Mathematics in America before 1900* (Chicago: Open Court Publishing Co., 1934), p. 91 ff.

a sailor and cooper, was himself apprenticed to a ship chandler at the age of ten after having attended school for only a few years. Although this ended his formal schooling, his studious habits led local men to allow him access to the books of the "Philosophical Library." This was one of two libraries financed by groups of private individuals in Salem. Its nucleus was the library of the Irish chemist and naturalist, Richard Kirwan, which had been a part of the booty taken from a British vessel in the Irish channel by an American privateer during the Revolution.[5] It was significant for Bowditch, and, through him, for American mathematics that this library contained continental as well as British works. Young Bowditch began to learn Latin by himself at the age of seventeen in order to read Newton's *Principia*. As we shall note shortly, one of his major achievements involved translating Laplace's *Mécanique Céleste* from the French. Struik says that he was "probably the first man in the United States who understood continental mathematics and who thus could break the spell of the exclusively English tradition in this field."[6] This interest in languages correlated well with both his interest in sailing and his interest in science. His chief tools in language study were New Testaments and dictionaries. His library contained New Testaments in twenty-five different languages, and he knew Latin, French, Spanish, Italian, Portuguese, and German, which latter he learned at the age of forty-five.

From 1795 to 1803 Bowditch sailed five voyages to China and the East Indies as clerk, supercargo, and finally as captain. During these voyages he studied French and the works of Lacroix and Laplace, and checked existing navigation charts and tables while learning old navigation techniques and improving on them. He found so many errors in the existing publications that he offered a compilation of them to Edmund March Blunt, a Newburyport newspaper publisher who also published Captain Furlong's *American Coast Pilot* as well as J. H. Moore's *The New Practical Navigator*. This latter was first published in London in 1772. Blunt's 1799 edition noted that it had been revised, corrected, and extended but did not mention Bowditch by name. However, in 1802 Blunt published *The New American Practical Navigator* with Bowditch's name as author and labeled "first edition" although in a preface he states that it would have been the third edition of Moore's book if there had not been too many errors in Moore. It has been stated that Bowditch found eight thousand errors and that one of them alone, the listing of 1800 as a leap year, had been responsible for many shipwrecks.[7]

This book became the seaman's bible. The fame of "Bowditch" paralleled that of its contemporaries, Nicholas Pike's arithmetic[8] and Noah Webster's dictionary. "Bowditch" appeared in ten editions during the author's life and after his death it was purchased from Blunt's sons by the United States Hydrographic Office. This office has continued to publish revisions of *The American Practical Navigator* up to the present day. "Bowditch" is still a navigator's bible, and pound for pound, one of the best book buys of today.

The first (1802) edition which I examined in the University of Michigan's William L. Clements Library contains chapters on decimal arithmetic, geometry, trigonometry, Gunter's Scale, and the description and use of the sector,[9] in addition to discussions and tables relating to maps and navigation. The section dealing with the solution of oblique triangles furnishes some ideas and procedures which might enrich teaching today.

[5] D. J. Stuik, *op. cit.*, p. 71.
[6] *Ibid.*, p. 72.

[7] *Ibid.*, p. 73.
[8] Fellow townsman Pike's book was discussed in THE MATHEMATICS TEACHER, XLVII (October 1954), pp. 409–410.
[9] For a recent discussion of the sector, see Florence Wood, "Tangible Arithmetic II: the Sector Compasses," THE MATHEMATICS TEACHER, XLVII (December 1954), pp. 535–542.

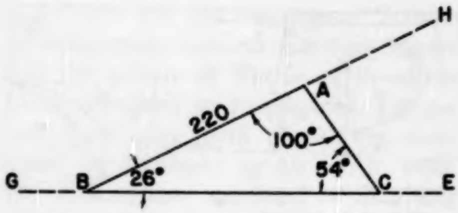


Figure 1

Each "case" is solved three ways: "by projection," "by logarithms," and "by Gunter" (using dividers and a Gunter scale). The three solutions for "Case I" on pages 68–69 are as follows: "Given (Fig. 1) $<BAC = 100°$, $<ACB = 54°$, $AB = 220$. Subtract the sum of angles A and C from 180° to find $<B = 26°$. Draw line GE and at any point B on it draw BH such that $<EBH = 26°$. Lay off $BA = 220$. On A make $<BAC = 100°$. AC intersects CE in C. Measure BC on the same scale as BA. $BC = 268$, nearly."

Bowditch's solution "by logarithms" is modern and is omitted here. It was based on "Theorem II" which is essentially the Sine Law. Theorem II was stated and proved on page 63 as follows: "In all plane triangles, the sides are in direct proportion to the sines of their opposite angles." The proof was based on similar triangles and the concept of the sine function as it was defined at that time; namely, a line equal to half the chord of twice a given arc in a circle of given radius. The ratios of two sines then would be the ratios of two such lines in the same circle or in circles with equal radii. The proof then was: make $AF = BC$ (Fig. 2). Then $FE = \sin A$, $BD = \sin C$ (In circles with radii $AF = BC$). Triangle ABD is similar to triangle AFE, and hence $AF:AB = FE:BD$ or $BC:AB = FE:BD$, which last equation is our theorem stated in symbols.

The solution "by Gunter" reads "The extent from the angle $C = 54°$ to the angle A or its supplement 80°, on the sines, will reach from $AB = 220$ to $BC = 268$ on the line of numbers." To interpret this one must visualize a Gunter scale such as was pictured in this department in the March 1955 issue of THE MATHEMATICS TEACHER. This scale has one line divided according to the logarithms of the sines of angles and a second line divided proportionally to the logarithms of numbers. Therefore if one picks up with a pair of dividers the distance between log sin C and log sin A on the *sine* scale, the points of the dividers really span a length equal to log (sin $C \div \sin A$). This length may be added on the number scale to the logarithm of the length of $AB = \log 220$ by putting one point of the dividers at 220 on the scale and reading at the other point the number corresponding to

$$\log AB + \log \frac{\sin C}{\sin A} \quad \text{or} \quad \log \frac{AB \sin C}{\sin A}.$$

There are many anecdotes to be told about Bowditch. One is that his enthusiasm for navigation and his passion for simplifying it and clarifying technical explanations led him to teach navigation to everyone in his crew so that at the end of his last voyage even the cook could determine the ship's position from celestial observations.

The same year that the first edition of *The American Practical Navigator* appeared, Harvard awarded Bowditch an honorary Master's degree. The following year he completed his last voyage, and in 1804 he became, at the age of thirty, the president of the Essex Fire and Marine Insurance Company. He administered this most successfully until 1823 when he moved to Boston to become Director of the Massachusetts Hospital Life Insurance

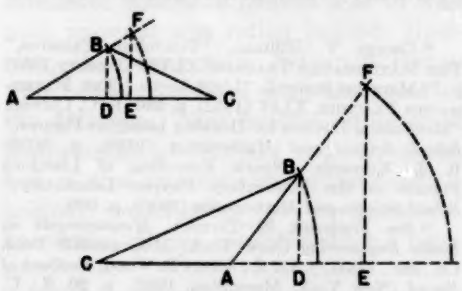


Figure 2

Company. During these years as he became a busy public figure, he was also a member of the American Academy of Arts and Sciences and continued his work on astronomy, navigation, and chart construction. He published thirty-one papers, mostly in the *Memoirs of the American Academy*, but also in the German *Astronomische Nachrichten* and other journals.

One of the most famous of these papers appeared in Volume III of the *Memoirs* in 1815. It was titled "On the Motion of a Pendulum Suspended from Two Points." Our Figure 3 shows the diagrams which accompanied this article. The pendulum is represented in the upper left hand corner. The curves in the other parts of the figure represent the paths of the pendulum as the result of varied initial conditions. Bowditch derived the equations

$$x = b \cos (at + c),$$
$$y = b' \sin (a't + c')$$

for these curves. These equations show that the curves include the conic sections and may be regarded as generated by combining two simple harmonic motions at right angles to each other. Various devices for drawing these curves have been described recently in this and other journals.[10]

One reason for this current interest in these curves, called Lissajous curves today, is their very practical usefulness in modern electronics and acoustics.[11] Bowditch's original interest, however, can be termed "practical" only by a considerable stretching of the term. He was motivated by a study (which had been begun by Pro-

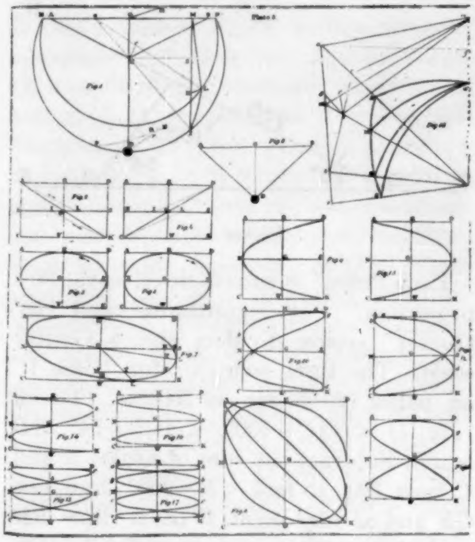


Figure 3

fessor James Dean of the University of Vermont) of the apparent motion of the earth as if viewed from the moon.

Later these curves were discovered independently by a French scientist, J. A. Lissajous, in connection with a study of vibrating strings.[12] Bowditch's earlier work was little known and as a result the curves are today most generally known by Lissajous' name.

Before passing to Bowditch's third major publication it is interesting to note that our Figure 3 is a photograph of a page in a reprint from the *Memoirs* sent by Bowditch himself to Thomas Jefferson, "late President of the United States." This and other reprints which had been received from various persons scattered throughout the world were bound together by Jefferson into a book for his library shelves. This book is now in the William L. Clements Library.

Bowditch's international reputation was based chiefly on his *Navigator* and on his translation of Laplace's *Mécanique Céleste*.

[10] George T. Hillman, "Pendulum Patterns," The Mathematics Teacher, XLVII (January 1954) p. 7; Margaret Sumrall, "Light Locus," The Mathematics Teacher, XLIV (1952), p. 586; R. C. Colwell, "Mechanical Devices for Drawing Lissajous Figures," *School Science and Mathematics* (1936), p. 1005; R. L. Edwards, "Spark Recording of Lissajous Figures in the Elementary Physics Laboratory," *School Science and Mathematics* (1930), p. 909.

[11] See Frederick E. Terman, *Measurements in Radio Engineering* (New York: McGraw-Hill Book Co., Inc., 1935), p. 327 ff.; Albert B. Wood, *Textbook of Sound* (New York: Macmillan, 1930), p. 20. ff.; C. M. Summers, "Production Testing," *General Electric Review*, 45 (1942), p. 702.

[12] J. A. Lissajous, "Mémoire sur le Position des Noeuds dans les Lames qui Vibrent Transversalement," *Annales de Chemie et de Physique*, series 3, vol. 30 (1850), pp. 385–410; *Étude Optique des Mouvements Vibratoires* (1873).

Our Figure 4 is the title page of Volume IV of this really monumental work. Figure 5 is the picture of Nathaniel Bowditch which served as its frontispiece. Volume four, containing 1018 pages, was published posthumously by his son in 1839. The first volume appeared in 1829 and contained 746 pages, volume two, 1832, contained 990 pages, volume three, 1834, contained 910 pages plus extensive tables. All of these pages were large $8\frac{1}{2} \times 11$ inches in size. Every page, with few exceptions, contained extensive footnotes and explanations by Bowditch. Frequently these notes occupied more than half the page. These were necessary because Laplace was notorious for his use of "Il est facile de voir" in the place of detailed expositions. J. B. Biot who assisted Laplace revise the book for printing said it often took the author himself an hour to rediscover the reasoning he had suppressed. Bowditch wrote "I never came across one of Laplace's 'This it plainly appears' without feeling sure that I had hours of hard work before me to fill up the chasm and to find out and show how it plainly appears."[13]

MÉCANIQUE CÉLESTE.

BY THE

MARQUIS DE LA PLACE,

PEER OF FRANCE; GRAND CROSS OF THE LEGION OF HONOR; MEMBER OF THE FRENCH ACADEMY, OF THE ACADEMY OF SCIENCES OF PARIS, OF THE BOARD OF LONGITUDE OF FRANCE, OF THE ROYAL SOCIETIES OF LONDON AND GÖTTINGEN, OF THE ACADEMIES OF SCIENCES OF RUSSIA, DENMARK, SWEDEN, PRUSSIA, HOLLAND, AND ITALY; MEMBER OF THE AMERICAN ACADEMY OF ARTS AND SCIENCES; ETC.

TRANSLATED, WITH A COMMENTARY,

BY

NATHANIEL BOWDITCH, LL. D.

FELLOW OF THE ROYAL SOCIETIES OF LONDON, EDINBURGH, AND DUBLIN; OF THE ASTRONOMICAL SOCIETY OF LONDON, OF THE PHILOSOPHICAL SOCIETY HELD AT PHILADELPHIA, OF THE AMERICAN ACADEMY OF ARTS AND SCIENCES; CORRESPONDING MEMBER OF THE ROYAL SOCIETIES OF BERLIN, PALERMO; ETC.

VOLUME IV.

WITH A MEMOIR OF THE TRANSLATOR,

BY HIS SON,

NATHANIEL INGERSOLL BOWDITCH.

BOSTON:

FROM THE PRESS OF ISAAC R. BUTTS;

CHARLES C. LITTLE AND JAMES BROWN, PUBLISHERS.

M DCCC XXXIX.

Figure 4

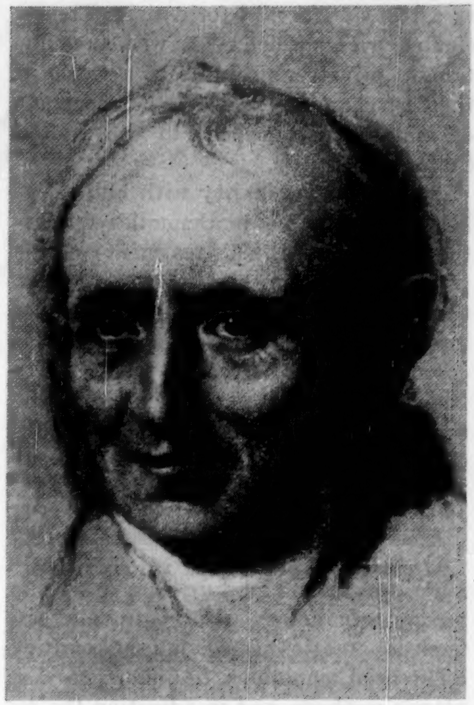


Figure 5

Nathaniel Bowditch from the frontispiece of Volume IV of his Mécanique Céleste. Compare this with the color reproduction of a painting by Charles Osgood on page 19 of *American Heritage*, Vol. VI, no. 2 (February, 1955).

Thus Bowditch's work was no mere translation, but contained considerable original material. Further it presented for the first time to American readers continental, as opposed to English, mathematics, and mathematics of a fairly advanced nature. Of course, the number of interested American readers able to read such material was rather limited. However, the young man, Benjamin Peirce, who helped Bowditch with the proof of the *Mécanique* later became professor of mathematics at Harvard and one of the first of this country's really outstanding original contributors to mathematical knowledge. Peirce used Bowditch's book

[13] D. E. Smith, *History of Mathematics* (Boston: Ginn and Co., 1923), vol. I, p. 487.

as a text at Harvard for a time. Thus a chain of events which began with a revolutionary privateer's capture of a vessel in the Irish channel effected the development of mathematical science in the United States.

Bowditch's familiarity with both English and continental mathematics was further revealed in his treatment of the pendulum problem mentioned earlier. In that problem, after resolving the forces into components parallel to rectangular coordinate axes, he stated "these forces being multiplied by the fluxion of the time dt will give the fluxions of the velocities in the direction of these axes namely

$$d\cdot\frac{dx}{dt},\quad d\cdot\frac{dy}{dt},\quad d\cdot\frac{dz}{dt},\ \ldots\text{"}$$

Here we see a mixture of Newtonian fluxion terminology with continental differential notation. In solving the resulting differential equations he refers to Laplace's methods, and in adjusting the results to accord with varied initial conditions he uses L'Hôpital's rule for indeterminate forms—without mentioning L'Hôpital however.

The financial problem of publishing such a huge work in a place where there was so little demand for it is revealed by the fact that the two additional volumes which Bowditch promised if there were sufficient demand never appeared, and by the fact that of the $36,571.38 estate left by Bowditch, $5,000.00 was the value placed upon the unsold copies of his book. The posthumous fourth volume contains a dedication to Mary Bowditch, his second wife and the mother of his six children, as well as a one hundred and sixty-five-page anecdotal biography by his son. The anecdotes reveal him as meticulously precise in his business conduct and relations with others. He was described as small-built, active, and prematurely gray. Something of the mental, financial and physical effort needed to produce the *Mécanique* can be read into the dedication of the book to "Mary Bowditch, who devoted herself to her domestic avocation with great judgment, unceasing kindness and a zeal which could not be surpassed; taking upon herself the whole care of her family and thus procuring for him (the author) the leisure hours to prepare the work; and securing to him, by her prudent management, the means for its publication in its present form, which she fully approved; and without her approbation the work would not have been undertaken."

Although Simon Newcomb, a later American mathematician and astronomer said that his work "betrays the want of that inspiration which comes from intimate contact with the masters," Struik says that Bowditch's *Mécanique* "provided the English speaking world at last with a full explanation of continental mathematics and theoretical astronomy, and at the same time brought the Newtonian period of American science to a climactic end."[14]

[14] Struik, *op. cit.*, pp. 75, 176.

"Though I am not and never was an editor, I know something of the trials to which they are submitted. They have nothing to do but to develop enormous calluses at every point of contact with authorship. Their business is not a matter of sympathy, but of intellect. They must reject the unfit productions of those whom they long to befriend, because it would be a profligate charity to accept them. One cannot burn his house down to warm the hands even of the fatherless and the widow."—*Oliver Wendell Holmes*, The Autocrat of the Breakfast-Table, *page* 294.

"What we have learned so far of the universe, both as a whole and in its microstructure, suggests that in neither aspect can it be treated merely as an enlarged or diminished version of the world which we know through our senses. The ultimate secrets of nature are written in a language which we cannot yet read. Mathematics provides a commentary on the text, sometimes a close translation, but in words we can read because they are our own."—*O. G. Sutton*, Mathematics in Action (London; *G. Bell and Sons*, 1954).

Edited by Paul C. Clifford, State Teachers College, Montclair, New Jersey, and Adrian Struyk, Clifton High School, Clifton, New Jersey

Alligation—a mathematical relic

by Kenneth C. Eveland, Union Hill High School, Union City, New Jersey

Mr. Gibbins' article, "Historical Extra Credit," in the November 1954 issue of THE MATHEMATICS TEACHER presented the following problem: "A grocer has coffee worth 8¢, 16¢ and 24¢ per pound respectively. How much of each kind must he use, to a cask holding 240 lbs., that shall be worth 20¢ a pound?" The answer given was a specific one: 40 @ 8¢, 40 @ 16¢ and 160 @ 24¢. This aroused my curiosity because it was evident that many answers were possible as long as the number of pounds of 24¢ coffee was more than 120 and less than 180. There are 58 different sets of answers if integers only are allowed, and an infinite number if fractions are permitted.

In his article the author explained that the problem was solved by "alligation," and this was, I must confess, an entirely new word to me. The dictionary was of little help. "Alligation: in old arithmetics a process or rule for the solution of problems concerning the compounding or mixing of ingredients differing in price or quality." This left me more puzzled than ever. What rule? What process?

In the belief that other teachers might be interested in this rule or process I have prepared this paper in which I hope these questions are answered. Much of the information here contained was found in *The University Arithmetic* published in 1863 and written by Charles Davies.

Alligation is the process of mixing substances in such a way that the value of the compound shall be equal to the sum of the values of the several ingredients. This process is divided into two classes—Alligation Medial and Alligation Alternate.

In Alligation Medial we find a method of determining the price or quality of a mixture of several simple ingredients whose prices and qualities are known. This is the easier type of alligation problem. For example: A grocer mixes 200 lbs. of one kind of sugar worth 8¢ a pound, 400 lbs. of another kind worth $9\frac{1}{2}$¢ a pound and 600 lbs. of a third kind worth 7¢ a pound. What should be the price of the mixture? The rule to be applied here is (1) find the cost of the mixture and (2) divide the cost of the mixture by the sum of the quantities. The quotient will be the price of the mixture. The solution is, of course, quite simple but I include it here as preparation for what follows:

200 Lbs. @ 8¢	=$16.00
400 Lbs. @ $9\frac{1}{2}$¢	= 38.00
600 Lbs. @ 7¢	= 42.00
1200 Lbs.	$96.00 (8¢ per lb.)

Alligation Alternate is the method of finding what ratios of several quantities, whose prices or qualities are known, must be taken to form a mixture of any required price or quality. It is the reverse of Alligation Medial, and may be proved by it. The process is founded on an equality of gain and loss. In selling a mixture at an average price there is a gain on each quantity below that price and a loss on each quantity

above that price. The gain must be exactly equal to the loss, otherwise the value of the compound would not be equal to the total value of the components.

There are three possible types of problems which may be solved by this method.

Type 1: A grocer wishes to mix sugar costing 12¢ a pound, sugar costing 8¢ a pound and sugar costing 5¢ a pound to make a mixture worth 7¢ a pound. What proportion of each should he use?

Solution

		A	B	C	D	E	
	5¢	½	½	5	1	6	3 lbs.
7¢	8¢		1		2	2	1 lb.
	12¢	⅕		2		2	1 lb.

On every pound put into the mixture whose price is less than the mean price there will be a gain, on every pound whose price is greater than the mean price there will be a loss; and since the gains and losses must balance each other, we must connect an ingredient on which there is a loss with one on which there is a gain. A pound of 5¢ sugar when put into the mixture will bring 7¢ or a gain of 2¢. Thus, to gain 1¢, we take half as much or ½ lb. This we write opposite 5¢ in column A. On a pound of 12¢ sugar there will be a loss of 5¢, so for a loss of 1¢ we take ⅕ of a pound. This is also written in column A, opposite 12¢. The numbers ½ and ⅕ we shall call proportional numbers. Comparing now the 5¢ sugar and the 8¢ sugar in the same way we write ½ in column B opposite 5¢ and 1 opposite 8¢. You will notice that each column contains one quantity on which there is a gain and one upon which there is a loss. If each time we take ½ lb. of 5¢ sugar we take ⅕ lb. of 12¢ sugar the gain and loss will balance; and if every time we take ½ lb. of 5¢ sugar we take 1 lb. of 8¢ sugar the gain and loss will balance. Therefore, if the proportional numbers in any column be multiplied by any number the gain and loss denoted by the products will balance. If these proportional numbers are fractional, as they are in this case, multiply them by the L. C. M. of their denominators and write the products in columns C and D. Now add the numbers in column C and D opposite each price, and if their sums have a common factor divide by it. The last result will be the proportional parts sought.

The solution given, i.e., 3 lbs., 1 lb. and 1 lb. is only one of an infinite number since, if the proportional numbers in any one column be multiplied by any number the gain and loss in that column will still balance, and the proportional parts in the final result will be changed.

The rules for this type of Alligation Alternate may be stated as follows:

1. Write the prices or qualities of the quantities in a column beginning with the lowest and with the mean price or quality at the left.
2. Opposite the first quantity write the part which must be taken to gain one unit of the mean price, and opposite the other quantity of that pair write the part which must be taken to lose one unit of the mean price. Do the same for each quantity.
3. When the proportional numbers are fractional, reduce them to integral numbers and then add those which stand opposite the same quantity; if the sums have a common factor, divide it out and the result will denote the proportional parts.

Type 2: A farmer wanted to mix tea worth 80 cents a pound, and tea worth 75 cents a pound with 66 pounds of tea worth 45¢ a pound so that the mixture would be worth 50¢ a pound. How much had to be taken of each sort?

Solution

		A	B	C	D	E	F
	45	1/5	1/5	6	5	11	66
50	75		1/25		1	1	6
	80	1/30		1		1	6

The solution here is the same as for type 1. But since we are to take 66 lbs. of 45¢ tea; each proportional number must be taken 6 times, which is the quotient of 66÷11. The rule here may be stated: (1) proceed as in type 1 and find the proportional parts, (2) find the ratio of the proportional part of the given quantity to the quantity

to be taken, and multiply each proportional part by it.

It is well to note that other solutions are possible. If we multiply either column C or column D by any other number, column E will be changed by it. For example, if we multiply column D by 12 the numbers in column E will become 66, 12, 1, and this solution satisfies the conditions of the problem.

Type 3: A jeweler has four sorts of gold—24 carats, 22 carats, 20 carats and 15 carats. He wishes to make a mixture of 42 ounces of 17 carats. How much must he take of each sort?

Solution

		A	*B*	*C*	*D*	*E*	*F*	*G*	*H*
	15	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	7	5	3	15	30
17	20			$\frac{1}{3}$			2	2	4
	22		$\frac{1}{5}$			2		2	4
17	24	$\frac{1}{7}$			2			2	4

The solution here is much the same as in type 2. The rule: (1) find the proportional numbers as in type 1, (2) divide the quantity of the mixture by the sum of its proportional parts; and the quotient indicates how many times each is to be taken. Multiply the parts separately by this quotient, and each product will indicate the amount of the corresponding quantity.

In the problem which Mr. Gibbins presented we should proceed as follows (a "type 2" problem):

		A	*B*	*C*	*D*	*E*	*F*
	8	$\frac{1}{12}$		1		1	40
20	16		$\frac{1}{4}$		1	1	40
	24	$\frac{1}{4}$	$\frac{1}{4}$	3	1	4	160

I conclude with two problems selected from *The University Arithmetic* which not only offer an example of Alligation Alternate but also provide a commentary on life in these United States circa 1860.

1. A tailor has 24 garments worth $144. He has coats, pantaloons and vests worth $12, $5, $2 each, respectively. How many has he of each? (Answer: 6 vests, 12 pantaloons, 6 coats.)

2. A man paid $70 to 3 men for 35 days' labor(!). To the first he paid $5 a day, to the second $1 a day, and to the third $½ a day: how many days did each labor? (Answer: 10 at $½; 15 at $1; 10 at $5.)

On certain cases of congruence of triangles[1]

by Victor Thébault, Tennie, Sarthe, France

It is the purpose of this note to present and to apply certain simple cases of congruence of triangles, obtained by associating sets of three corresponding elements.

I. *Theorem. Two triangles ABC, A′B′C′ are congruent if* $AB=A'B'$, $AC=A'C'$, *and* $AM=A'M'$, *where points M and M′ divide BC and B′C′ so that*

$$BM/MC=B'M'/M'C'=1/k$$

Proof: As in Figure 1, let the lines through C and C' parallel to AB and $A'B'$ meet lines AM and $A'M'$ at N and N'. By the theorem of Thales

$$\frac{AM}{MN}=\frac{AB}{NC}=\frac{BM}{MC}=\frac{1}{k}.$$

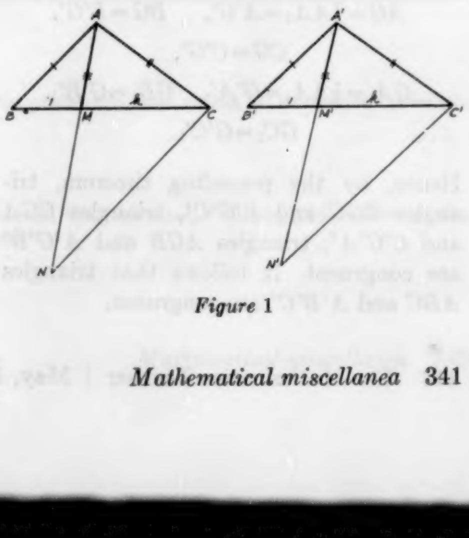


Figure 1

[1] Translated from the French by Adrian Struyk.

Hence

$$\frac{AM}{AM+MN}=\frac{AM}{AN}=\frac{1}{1+k},$$

or

$$AN=(1+k)AM.$$

Similarly, $A'N'=(1+k)A'M'$, so that $AN=A'N'$. Also $NC=N'C'$, since $NC=k\cdot AB=k\cdot A'B'=N'C'$. Consequently, triangles ANC and $A'N'C'$ are congruent, having three sides of one equal to three sides of the other. Then

$$\angle NAC=\angle N'A'C',$$

and

$$\angle NAB=\angle N=\angle N'=\angle N'A'B'.$$

Therefore triangles ABC and $A'B'C'$ are congruent, since the pairs of equal sides now include equal angles.

Particular cases. With the same hypothesis $AB=A'B'$, $AC=A'C'$ the triangles ABC, $A'B'C'$ are congruent if the segments AM, $A'M'$ are identified with the medians, the symmedians, the interior angle-bisectors, or the exterior angle-bisectors. In these cases the ratio $1/k$ has the respective values 1, $(AB/AC)^2$, AB/AC, AB/AC.

II. *Applications. Two triangles ABC, $A'B'C'$ are congruent if the medians of one are equal, respectively, to the medians of the other.*

Proof: If the medians AA_1, BB_1, CC_1, which are concurrent at G, are respectively equal to $A'A_1'$, $B'B_1'$, $C'C_1'$, which are concurrent at G', then

$$AG=\tfrac{2}{3}AA_1=A'G',\qquad BG=B'G',$$
$$CG=C'G',$$
$$GA_1=\tfrac{1}{3}AA_1=G'A'_1,\quad GB_1=G'B'_1,$$
$$GC_1=G'C'_1.$$

Hence, by the preceding theorem, triangles BGC and $B'G'C'$, triangles CGA and $C'G'A'$, triangles AGB and $A'G'B'$ are congruent. It follows that triangles ABC and $A'B'C'$ are congruent.

N.B. By extending GA_1 and $G'A_1'$, each its own length to N and N', the congruence of triangles GNC and $G'N'C'$, and hence the congruence of BGC and $B'G'C'$, $\cdots$, ABC and $A'B'C'$, is established by using only Book I propositions of geometry.

Two tetrahedra $ABCD$, $A'B'C'D'$ are congruent if three medians and the three bimedians of one are respectively equal to the corresponding elements of the other.

Proof: The line-segments AG_a, BG_b, CG_c, DG_d (the medians) which join the vertices A, B, C, D of tetrahedron T to the centroids of the opposite faces BCD, CDA, DAB, ABC, and the line-segments A_1A_1', B_1B_1', C_1C_1' (the bimedians) which join the midpoints of opposite edges BC and DA, CA and DB, AB and DC, all concur in the centroid G of T. Now G bisects each bimedian, and divides each median into segments with ratio 1 to 3. That is,

$$G_aG=\tfrac{1}{4}G_aA. \qquad \text{(See Fig. 2.)}$$

Let those medians of T which issue from vertices, B, C, D be respectively equal to the corresponding medians of a tetrahe-

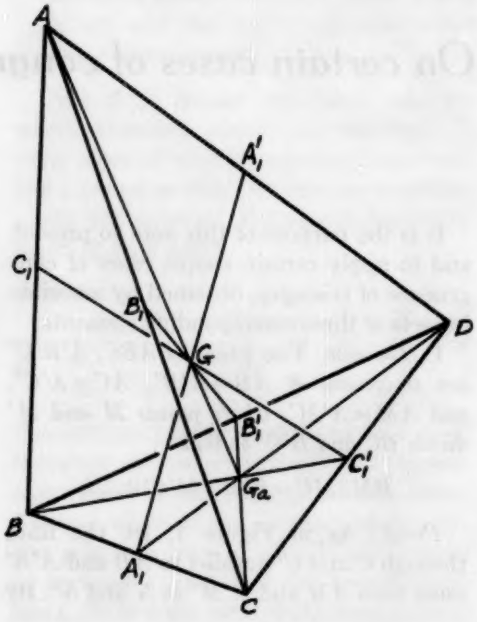


Figure 2

dron T', and let the corresponding bimedians of T and T' be equal each to each. Then from theorem I follows the congruence of triangles GBC and $G'B'C'$, of GCA and $G'C'A'$, of GAB and $G'A'B'$, for each pair has two sides and the included median of one triangle equal to two sides and the included median of the other. An immediate consequence is the congruence of triangles BCD and $B'C'D'$ ($sss = sss$), and then in turn the congruence of the tetrahedra $GBCD$ and $G'B'C'D'$, which have their faces congruent, each to each. Hence,

$$GG_a = G'G'_a.$$

As a result of this we may write

$$AG_a = 4 \cdot GG_a = 4 \cdot G'G'_a = A'G'_a.$$

That is, all four medians of T and T' are equal, each to each. That the remaining faces of T and T' are congruent, each to each, can now be proven by the same method used for BCD and $B'C'D'$. The tetrahedra T and T' are therefore congruent.

Exercises

1. What can be said of two triangles ABC and $A'B'C'$ which have sides $AB = A'B'$, $AC = A'C'$, and altitudes $AM = A'M'$?
2. Two tetrahedra are congruent if the medians and two bimedians of one are equal to the corresponding elements of the other.
3. Two triangles are congruent if their altitudes are equal, each to each.

When High School Is Out

Most boys and girls burst out of doors;
But there are others, who,
Though deep in thought, but not on books,
Walk, two by two.

These are reluctant to go home,
But, hand in hand, surmise
Alluring lessons to be read
In others' eyes.

They wander as through poets' rhymes;
Till, in the golden weather,
Instructions not in books will bring
Their lips together.

—*Louis Ginsberg*, The Clearing House *March*, 1955

Teaching is the profession that makes all professions possible.

"One who enters teaching enters a career that offers all the freshness, vitality, and challenge of a rapidly developing profession."—*Benjamin W. Frazier*

Edited by Lucien B. Kinney, School of Education, Stanford University, and Dan T. Dawson, School of Education, Stanford University, Stanford, California

Remedial work in junior high school mathematics

by Harl R. Douglass, Director of College Education, University of Colorado, Boulder, Colorado

Because of the very nature of mathematics learning the instructional materials in that field are necessarily sequential. While perhaps a greater degree of flexibility exists in seventh and eighth grades than in earlier grades, there is much material in seventh- and eighth-grade mathematics which can hardly be learned at all unless appropriate background concepts, principles, and skills in mathematics have been acquired in previous grades. In addition to these mathematical learnings there are also understanding, readiness, attitudes, and skills in problem solving, which are essential to success in the junior high school.

While there have always been pupils in the seventh and eighth grades with inadequate elementary preparation, the situation has become worse in recent years. For some twelve or fifteen years now, there has been an increased tendency to promote every youngster a grade a year with little reference to the degree of achievement in arithmetic, reading, language arts, and other subjects. The majority of pupils entering junior high school have come up from elementary schools employing that practice, which means, of course, that a considerable proportion of the youngsters in the seventh grade are not better in arithmetic than the average sixth-grade pupils. Some are, in fact, no better than the average fifth-grade pupil, and there is occasionally a youngster who is not better than the average fourth-grade pupil. It is quite obvious that pupils with such deficiencies have little hope of learning the conventional seventh-grade mathematics without considerable review and very definite rehabilitation in the fundamental processes and skills.

The need for such remedial work accounts for the fact that in the 1920's, and particularly in the 1930's, homogeneous grouping was widely employed in the junior high schools. Indeed, it was the exceptional junior high school that did not have homogeneous grouping in some form or another. In the late 1930's, homogeneous grouping was abandoned in some schools, and throughout the 1940's the number of schools in which it was abandoned steadily increased until today homogeneous grouping is employed, according to a recent survey made by the author, in slightly less than half of the junior high schools.

Homogeneous grouping began to disappear for several reasons. In the first place, in grouping practices, not the best methods were employed. In addition, some teachers felt the problem of adjusting to individual differences would be magically solved by grouping, and little

attention was given to adapting learning activities and materials to the individual. Accordingly, after the youngsters were grouped, instruction was not appropriately adapted to their capacities, abilities, present status of achievement, needs, and interests. Furthermore, teachers who had the background, training, and/or experience with the very slow and the very bright were not assigned to the appropriate classes. As a result, the situation had all the weaknesses of grouping and little or none of the advantages. Moreover, parents objected strenuously to having their youngsters labeled as dull normals. Others, on general grounds, protested loudly that these dull youngsters were stigmatized and that the procedure was undemocratic.

All these considerations focused attention on individualized instruction, with the claim that adaptation to the individual could be better handled in the heterogeneous class. This hope does not seem to be supported in practice today. It is one of those attractive sounding theories that just does not always work out in practice, probably because the teachers do not understand the problem of adapting learning activities and materials to the very slow or the very bright while at the same time handling the large middle level. Even more probably, it may be attributed to the fact that teachers have not sufficient time to give adequate preparation and planning, and that in classrooms they are too busy to give attention to the individuals or small groups of individuals at either end of the spectrum of abilities and background. In any case, the problem has not been solved. In many schools the situation is about what it was in 1919, and in most schools it is very little better.

In recent years the aggravation of the problem has caused the number of schools employing some type of grouping in seventh and eighth grade to increase. This is particularly true with respect to the setting up of special sections for youngsters who need instruction and preparation in the arithmetic of grades four, five, and six before attempting the new materials in grades seven and eight. The same sort of situation is found in reading; in most junior high schools at least one section of remedial reading has been set up in the seventh and also in the eighth grade. In these schools the students are reported as getting along at least as well as they would in the heterogeneous groups and in a satisfying number of cases the situation seems to be very materially improved.

Experiences of the 1920's and 1930's point clearly to the conditions necessary if these special sections are to be a success. Special attention needs to be given both to determining which of the students are to form the special class, and to the selection and use of instructional materials and instructional and learning activities.

The selection of youngsters to constitute the slow section needs to be carefully done on the basis of several factors. Most significant of these is the score on an achievement test covering fifth- and sixth-grade arithmetic. The students who are obviously weak in fifth- and sixth-grade arithmetic are, very likely, prospects for the special section. However, certain other factors need to be considered too. The student who is rather weak in fifth- and sixth-grade arithmetic, but who has a fairly high I.Q. (let us say 105 or higher) might well be classified in a heterogeneous section after the parents and the pupil have been fully acquainted with the situation in a conference with teacher and counselor, and there seem to be some prospects that adequate time and attention will be given by the pupil to the problem of catching up with his fellows. To such pupils, teachers must give special attention, motivation, and instruction.

The selection of a teacher for the "remedial" section is of great importance. He must not only have a real interest in assisting this type of youngster, but also have considerable ingenuity and creative ability in the selection and arrangement of instructional and learning materials.

While mathematical processes and concepts to be taught are those of the fifth and

sixth grades, and may even be to some extent those of the fourth grade, they must be taught in connection with problems that are appropriate to the interests and the maturity of seventh graders. The instructional activities must be appropriate to those employed with youngsters of their age and maturity. In other words, the mathematical processes must be those of lower grades, but the content of their application and the method of instruction must be that of the seventh grade. Thus one cannot use either the seventh- and eighth-grade textbooks or the fifth- and sixth-grade textbooks as basic texts in this class unless some departure and supplementation is planned.

The question arises as to whether or not these youngsters will be rehabilitated in a year and go into heterogeneous sections of the eighth grade. Usually it is found that they will not, although the hope should be held out to them that it is a possibility. Perhaps it should not be thought of as a promise nor very much emphasized in the minds of the youngsters. For the great majority of the youngsters in the special section, an eighth-grade special section must be formed the following year. Those in the special section of the eighth grade will do well if they are ready to go on with seventh-grade arithmetic. If there are not enough of them to form a full section in the eighth grade it may be that they should be permitted to go into the seventh-grade heterogeneous section. If there are enough of them, and the appropriate teacher can be employed, a considerable amount of the material from both grades seven and eight might be taught them in their eighth-grade special section, making certain that the more difficult processes and the difficult and not widely used topics, such as compound interest, square root, the metric system, etc., may be omitted.

Unfortunately, appropriate textbooks are not available for such classes, and there must be much special planning of work by the teacher, with preparation of a great deal of mimeographed material in the nature of worksheets. Even if good textbooks were available, a great deal of ingenuity and creative ability on the part of the teacher must be employed in making the work not only attractive and appealing but quite concrete and easily understood for the individual pupil.

While setting up the special section should relieve the seventh-grade and eighth-grade teachers of great demands on their time, both in class and out, and in dealing with individuals and in planning for them, teachers still have the responsibility of adapting to the individuals in the heterogeneous classes and the impression that their responsibility is greatly lessened should not be permitted to develop. Indeed, it is increased because, while they are now relieved of the most time-demanding slow youngster, they have an increased opportunity, and, therefore, responsibility, of doing something to see that the bright youngsters are properly challenged.

"In this connection, it is important to realize that social scientists believe that learning is better advanced through reinforcement and reward than through challenge, whereas in our mathematics courses, we tend to challenge the student. Here, again, we need better mathematics teaching. It is worth noting that the great interest of mathematicians in the type of curriculum we suggest may well produce a higher level of teaching than now exists."—*W. G. Madow, "Mathematics for the Social Scientists,"* The American Mathematical Monthly (October 1954)

• MEMORABILIA MATHEMATICA

Edited by William L. Schaaf, Brooklyn College, Brooklyn, New York

A. N. Whitehead on mathematical education

A quarter of a century has slipped by since the late Alfred North Whitehead[1] published *The Aims of Education.* Much has happened in the world since then, including the world of education. Yet this great philosopher's observations on mathematical education are as fresh and as pertinent today as when they were written. It was the "unfitness of recondite subjects for use in general education" which was of grave concern to Whitehead, who wrote:

It is in fact rather a delicate subject for mathematicians. Outsiders are apt to accuse our subject of being recondite. Let us grasp the nettle at once and frankly admit that in general opinion it is the very typical example of reconditeness. By this word I do not mean difficulty, but that the ideas involved are of highly special application, and rarely influence thought.

This liability to reconditeness is the characteristic evil which is apt to destroy the utility of mathematics in liberal education. So far as it clings to the educational use of the subject, so far we must acquiesce in a miserably low level of mathematical attainment among cultivated people in general. I yield to no one in my anxiety to increase the educational scope of mathematics. The way to achieve this end is not by a mere blind demand for more mathematics. We must face the real difficulty which obstructs its extended use.

Is the subject recondite? Now, viewed as a whole, I think it is. *Securus judicat orbis terrarum*—the general judgment of mankind is sure.

The subject as it exists in the minds and in the books of students of mathematics is recondite. It proceeds by deducing innumerable special results from general ideas, each result more recondite than the preceding. It is not my task this afternoon to defend mathematics as a subject for profound study. It can very well take care of itself. What I want to emphasize is, that the very reasons which make this science a delight to its students are reasons which obstruct its use as an educational instrument—namely, the boundless wealth of deductions from the interplay of general theorems, their complication, their apparent remoteness from the ideas from which the argument started, the variety of methods, and their purely abstract character which brings, as its gift, eternal truth.

Of course, all these characteristics are of priceless value to students; for ages they have fascinated some of the keenest intellects. My only remark is that, except for a highly selected class, they are fatal in education. The pupils are bewildered by a multiplicity of detail, without apparent relevance either to great ideas or to ordinary thoughts. The extension of this sort of training in the direction of acquiring more detail is the last measure to be desired in the interests of education.

The conclusion at which we arrive is, that mathematics, if it is to be used in general education, must be subjected to a rigorous process of selection and adaptation. I do not mean, what is of course obvious, that however much time we devote to the subject the average pupil will not get very far. But that, however limited the progress, certain characteristics of the subject, natural at any stage, must be rigorously excluded. The science as presented to young pupils must lose its aspect of reconditeness. It must, on the face of it, deal directly and simply with a few general ideas of far-reaching importance.

The remainder of this stimulating essay on "The Mathematical Curriculum" spells out to some extent what these "few general ideas of far-reaching importance" are, and how they might well be treated.

In Whitehead's opinion, the main ideas which lie at the base of mathematics are not at all recondite, although they are abstract. He considers the three interconnected groups of relations—relations between numbers, between quantities, and space relations—as the essence of mathematics from the standpoint of general education. The "fatal reconditeness" can be avoided by centering attention upon a few basic concepts, and not accumulating a host of details. Among these "main

[1] A. N. Whitehead, *The Aims of Education and Other Essays* (New York: The Macmillan Company, 1929; also, Mentor Books, The New American Library, 1949). Quoted by permission.

ideas" he would include the formulation of physical laws, the principles of numerical measurement, the function concept, the elements of logical method, and some understanding of the significance of the history of mathematics.

On equipment and facilities for mathematical research . . . .

We hear a great deal these days about visual aids and other equipment for learning and teaching mathematics. In this connection, the following letter may be of interest to our readers. It was written nearly ninety years ago by a young mathematics teacher at Oxford University, in reply to a letter suggesting new facilities for mathematical research.

DEAR SENIOR CENSOR:

In a desultory conversation on a point connected with the dinner at our high table, you incidentally remarked to me that lobster-sauce, "though a necessary adjunct to turbot, was not entirely wholesome."

It is entirely unwholesome. I never ask for it without reluctance: I never take a second spoonful without a feeling of apprehension on the subject of possible night-mare. This naturally brings me to the subject of Mathematics, and of the accommodation provided by the University for carrying on the calculations necessary in that important branch of Science.

As Members of Convocation are called upon (whether personally, or, as is less exasperating, by letter) to consider the offer of the Clarendon Trustees, as well as every other subject of human, or inhuman, interest, capable of consideration, it has occurred to me to suggest for your consideration how desirable roofed buildings are for carrying on mathematical calculations: in fact, the variable character of the weather in Oxford renders it highly inexpedient to attempt much occupation, of a sedentary nature, in the open air.

Again, it is often impossible for students to carry on accurate mathematical calculations in close contiguity to one another, owing to their mutual interference, and a tendency to general conversation: consequently these processes require different rooms in which irrepressible conversationalists, who are found to occur in every branch of Society, might be carefully and permanently fixed.

It may be sufficient for the present to enumerate the following requisites: others might be added as funds permitted.

A. A very large room for calculating Greatest Common Measure. To this a small one might be attached for Least Common Multiple: this, however, might be dispensed with.

B. A piece of open ground for keeping Roots and practicing their extraction: it would be advisable to keep Square Roots by themselves, as their corners are apt to damage others.

C. A room for reducing Fractions to their Lowest Terms. This should be provided with a cellar for keeping the Lowest Terms when found, which might also be available to the general body of under-graduates, for the purpose of "keeping Terms."

D. A large room, which might be darkened, and fitted up with a magic lantern, for the purpose of exhibiting Circulating Decimals in the act of circulation. This might also contain cupboards, fitted with glass-doors, for keeping the various Scales of Notation.

E. A narrow strip of ground, railed off and carefully leveled, for investigating the properties of Asymptotes, and testing practically whether Parallel Lines meet or not: for this purpose it should reach, to use the expressive language of Euclid, "ever so far."

This last process, of "continually producing the Lines," may require centuries or more: but such a period, though long in the life of an individual, is as nothing in the life of the University.

As Photography is now very much employed in recording human expressions, and might possibly be adapted to Algebraical Expressions, a small photographic room would be desirable, both for general use and for representing the various phenomena of Gravity, Disturbance of Equilibrium, Resolution, etc., which affect the features during severe mathematical operations.

May I trust that you will give your immediate attention to this most important subject?

Believe me,
Sincerely yours,
MATHEMATICUS.

February 6, 1868.

The writer[2] of this letter, in case you have not already guessed, was Charles Lutwidge Dodgson, and the several things he suggests for assisting the calculations all had their serious counterpart.

Early teachers of mathematics

Today, if one wishes to learn mathematics, there are ample opportunities in the schools, colleges and universities throughout the land. One rarely seeks such instruction from private tutors, except under unusual circumstances. But in the

[2] By special permission of G. P. Putman's Sons; from *Logical Nonsense, The Works of Lewis Carroll,* edited by Philip C. Blackburn and Lionel White, 1934.

seventeenth and eighteenth centuries it was otherwise. There were, for example, many outstanding mathematical lecturers in English universities, notably at Cambridge, as well as in universities scattered over the Continent. One recalls men like Isaac Barrow, a forceful and popular lecturer at Cambridge who voluntarily relinquished the Lucasian chair of mathematics to his pupil Isaac Newton, whose superior abilities he was quick to recognize and frank enough to acknowledge. Another well-known teacher at Cambridge was the blind lecturer in optics, Nicholas Saunderson. And there were many others, such as John Wallis of Oxford, Cavalieri of Bologna, and Gauss of Göttingen. Not infrequently mathematics was taught by well-known scholars under the patronage of royalty. Euler, the Swiss mathematician, was invited to Russia by the Empress, and later brought to the court of Frederick the Great. Descartes, the French soldier-philosopher-mathematician, was in later life called to the court of Sweden by Queen Christine, a headstrong young woman who insisted that he tutor her daily in philosophy at five o'clock in the morning.

More interesting, perhaps, was the rather common practice of teaching elementary mathematics privately. These teachers were, for the most part, mathematicians of lesser note; in some cases, self-styled mathematicians. Many of them were authors of textbooks. The following is a reproduction of a typical eighteenth-century advertisement of a private teacher:

> MATHEMATICKS, with their Ufe and Application to Natural Philofophy, Taught by the Author, at the *Hand and Pen* in *Little-Moorfields*: Where are taught Writing, Arithmetick, and Merchants Accompts; Alfo Youth Boarded, by *Ralph Snow.*

Here is another example, equally naive:

> At the *Hand* and *Pen* in *Barbican,* are Taught, *viz.* Writing in all Hands, Merchant's Accounts, Book-keeping, Algebra, Geometry, Measuring, Surveying, Gauging, Mechanicks, Fortification, Gunery, Navigation, Dialling, and other Parts of Mathematicks; also the Use of the Globes and Maps, after a Natural, Easy, and Concise Method, without Burthen to the Memory.

One of the most colorful of these early self-styled teachers was a William Leybourn, who wrote more than twenty textbooks. One of these, published in 1694, called *Pleasure with Profit,* I have already alluded to in an earlier issue. Among the "arts and sciences mathematical professed and taught" by the author were "arithmetick, geometrie, astronomie, and, upon these foundations, the following superstructures: the use of geometrical instruments, trigonometria, navigation, and horologiographia, or Dialling." Note the absence of any allusion to algebra. His advertisement was frank enough:

> The Place of the Author's Residence is about Ten miles from *London* Westward, at a Place called Southal, in the Road between *Action* and *Uxbridge,* and Three miles from *Brainford:* Where he intends to *Read* the *Mathematicks,* and Instruct young Gentlemen, and others: And to Board upon reasonable Terms, all such as shall be pleased to make a more close Application to those Studies: Where such Boarders, and others, (during their time of Residence with him) shall have the Use of all *Books, Maps, Globes* and other *Mathematical Instruments,* as are necessary for their Instruction, till they provide themselves of such as they shall have occasion for afterwards.

A further sidelight on this vogue of private teaching is revealed by the observation of William Webster (1740), himself a teacher, and author of *Arithmetick in Epitome.* Said he:

> When a Man has tried all Shifts, and still failed, if he can but scratch out anything like a fair Character, tho' never so stiff and unnatural, and has got but *Arithmetick* enough in his Head to compute the number of Minutes in a Year, or the Inches in a Mile, he makes his last Recourse to a Garret, and, with the Painter's Help, sets up for a Teacher of *Writing* and *Arithmetick;* where, by the Bait of low Prices, he perhaps gathers a Number of Scholars.

It must not be inferred, however, that all such private teaching was of an ele-

mentary sort or of an inferior grade. An outstanding exception was the skillful and scholarly instruction of William Oughtred, a great 17th century teacher of mathematics. Oughtred's chief concern was the ministry. But his love for mathematics (for which he had considerable aptitude) and his resourcefulness in making mathematical instruments consumed much of his time and energy. He gave private lessons, without remuneration, to pupils genuinely interested in mathematics. Among his noted pupils were John Wallis, whose book, *Arithmetic of Infinites*, was one of the forerunners of the calculus; Sir Christopher Wren, the famous architect who designed St. Paul's Cathedral in London; and Seth Ward, the celebrated astronomer. Oughtred was a staunch advocate of the use of symbols in mathematics. He used more than 150 symbols, most of which he invented. Only three of these have survived to modern times, namely, $\times$ for multiplication, :: for proportion, and $\sim$ for difference; of these, the last two have almost completely fallen into disuse. Oughtred is also to be remembered for his invention of a rectilinear and a circular slide rule. His most famous book, the *Clavis Mathematicae*, or *Key to the Mathematicks*, was the outgrowth of a manuscript prepared for the instruction of one of his pupils. This popular book, which first appeared in 1631, passed through many editions. It was probably the most influential mathematical book published in England for nearly fifty years from the time that Napier published his treatise on logarithms.

While on the subject of early teachers of mathematics, a word about early arithmetic books may also be of interest. Perhaps the reader shares our penchant for delving into books of bygone days. If so, he will understand the pleasures of the antiquarian. Not the least rewarding is an examination of arithmetic textbooks of several hundred years ago. The contrast with modern schoolbooks is almost unbelievable. One wonders how anyone could actually learn arithmetic with some of these old books. Among some of the features of these early arithmetics which will impress the modern reader are the naive definitions, the quaint language, the strange and obsolete weights and measures once in common use, the utterly unrealistic and whimsical problems, the frequent use of verse, and the vanity of many of the authors, to say nothing of their craving for publicity.

The very titles themselves have preserved the flavor of the times in which they were written. We reproduce herewith a few of them for your perusal:

The Well Spryng of Science, which teacheth the perfecte woorke and practise of Arithmeticke—beautified with moste necessary rules and questions.

Humfrey Baker; **London, 1568**

The arte of vulgar arithmeticke, both in integers and fractions, devided into two bookes—whereunto is added a third book entituled *Musa Meratorum*, comprehending all rules used in the most necessarie and profitable trade of merchandise—Newly collected, digested, and in some part devised, by a wel willer to the Mathematicals.

Thomas Hylles; **London, 1600**

The Hand-Maid to Arithmetick refined: Shewing the variety and facility of working all Rules in whole Numbers and Fractions, after most pleasant and profitable waies. Abounding with Tables above 150. for Monies, Measures and Weights, tale and number of things here and in forraigne parts; verie usefull for all Gentlemen, Captaines, Gunners, Shopkeepers, Artificers, and Negotiators of all sorts: Rules for Commutation and Exchanges for Merchants and their Factors. . . .

Nicholas Hunt; **London, 1633**

$$\text{A}\qquad \left.\begin{matrix}\text{Platform}\\ \text{Guide}\\ \text{Mate}\end{matrix}\right\}\ \text{for}\ \left\{\begin{matrix}\text{Purchasers}\\ \text{Builders}\\ \text{Measurers}\end{matrix}\right.$$

William Leybourn; **London, 1668**

Cocker's Arithmetick: Being A plain and familiar Method, suitable to the meanest Capacity for the full understanding of that Incomparable Art, as it is now taught by the ablest School-Masters in City and Country. Composed by Edward Cocker, late Practitioner in the Arts of Writing, Arithmetick, and Engraving. Being that so long since promised to the World.

Edward Cocker; **London, 1700**

Arithmeticke Made Easie for the Use and Benefit of Trades-Men.

Joh. Ayres; **London, 1714**

An Essay to facilitate Vulgar Fractions; after a

New Method, and to make Arithmetical Operations Very Concise.
Wm. Bridges; London, 1718

The Schoolmasters Assistant: being a Compendium of Arithmetic, both Practical and Theoretical—The Whole being delivered in the most familiar Way of *Question* and *Answer*, is recommended by several eminent *Mathematicians*, *Accomptants*, and *Schoolmasters*, as necessary to be used in *Schools* by all Teachers, who would have their *Scholars* thoroughly understand, and make a quick Progress in ARITHMETIC. To which is prefixt, an ESSAY on the *Education* of YOUTH; humbly offer'd to the Consideration of PARENTS.
Thomas Dilworth; London, 1762

An Introduction to so much of the Arts and Sciences, More immediately concerned in an Excellent Education for Trade In its lower Scenes and more genteel Professions.
J. Randall; London, 1765

An Essay on Arithmetic—Briefly, Shewing, First, the Usefulness; Secondly, It's extensiveness; Thirdly, The Methods of it.
William Wallis; London, 1790 (?)

The Young Arithmetician's Assistant; Comprising a Select Compilation of useful Examples in the Minor Rules of Arithmetic, Simplified and Adapted to the capacities of Young Ladies and Gentlemen composing the Junior Classes in Academical Establishments.
H. Marlen & W. Smeeth; Canterbury, 1823

A New System of Mental Arithmetic, by the Acquirement of which all numerical questions may be promptly answered without recourse to pen or pencil, &.
Daniel Harrison; London, 1837

On determining the area of a mathematician. . . .

Among interesting works that have come down to us from eighteenth century England there is to be found the *Ladies' Diary*, an annual publication in the form of an *Almanack*. In addition to enigmas, rebuses and other material, it contained many "Mathematical Questions," proposed by interested readers, and their answers as given by the proposers or other readers, often under a "feigned name." This remarkable publication flourished from 1704 to 1816, under the aegis of half a dozen different editors; during the last forty years or more of its existence it was ably directed by Dr. Charles Hutton, for many years professor of mathematics at the Royal Military Academy of Woolwich, and author of a *Mathematical and Philosophical Dictionary*. In 1817, one Thomas Leybourn, of the Royal Military College, republished all the mathematical questions and their solutions in a work comprising four volumes.

The following piece, a so-called "prize question," was proposed in the *Diary* in 1751 by a contributor using a Greek pseudonym.[3] After recalling the story of Archimedes running from his bath crying "Eureka!", our unknown proposer states his query:

> It is proposed to determine by the best method, the nearest superficial content in inches of a modern mathematician, of a middle age, weighing 160 pounds avoirdupois, being naked, all his parts middle-sized, and meanly proportioned; and his muscles not rigidly swelled, nor yet quite unbraced?
>
> *N.B.* The same rule will hold good for male or female mensuration; and man and woman being microcosms, expressions of many elegant and useful curves may thence be discovered; and several improvements made in the rectifications of curve lines, and quadrature of curvilinear spaces; besides cubation of several important solids; whose forms of fluxions, with their fluents, we shall insert in our new *Harmonia Mensurarum*.
>
> *Answered by* Mr. F. Holden, *at Westhouse, near Settle, Yorkshire.*
>
> Take a piece of wood or a stone, of known superficies, and, dipping it into a vessel full of melted tallow, you may, by trying the weight of the tallow and dipping-vessel in a scale before and after dipping, know the quantity of tallow, in weight, taken up by the piece of wood or stone. Then take about 18 or 20 modern mathematicians, (the more the better) strip them stark-naked, and suspend them (like Absoloms) by the hair of their heads, as chandlers hang their candles, or else by soft bandages under the chin and behind the nape of the neck, so that they may be raised or led down by pullies without hurting; dip them also, one by one, in the same vessel where the wood or stone was lately dipped, and mark the tallow they all take up, by weighing the vessel and tallow, before and after they are all dipped, (keeping the tallow just melted and of an equal warmth); Then say, as the quantity of tallow, in weight, taken up by the wood or stone, is to the known superficies of either, so is the weight of tallow taken up by all the mathematicians to the superficies of all the mathematicians. But, by all means, take

[3] Thomas Leybourn, *The Mathematical Questions, Proposed in the Ladies' Diary* (London, 1817), Vol. II, p. 71.

care that they are kept naked till they are shivering, and almost as cold as the wood or stone itself, before they are dipped, else this proportion will not hold good.

When they are all dipped, well scoured with soap, and cleansed from the tallow, let them be weighed, (or they may be all weighed before dipping) and say, as the weight of them all in pounds is to the late-found superficies of them all in square inches or square feet, so is 160 pounds weight to the superficies of the modern mathematician required to be known, ($=14\frac{1}{2}$ square feet, nearly, as we find by another method).

Letters to the editor

Dear Sir,

I was a little puzzled by one item in the January 1955 issue, "Erroneous arithmetical notions" on page 25. Is the title supposed to refer to De Morgan's ideas or the ideas he is quoted as criticizing? It seems to me that De Morgan is as outdated on this point as are some of the old ideas on imaginary numbers. All arithmetical operations are applied to numbers. When no units are attached, numbers are sometimes called "abstract"; and when units are attached, they are sometimes called "concrete." The terminology is unfortunate, since a number is just a number regardless of the units associated, with it. Any and all arithemetic operations can be carried out on numbers regardless of whether units are attached and regardless of what units are attached. Multiplication may sometimes be interpreted in terms of repetition, but this is not essential. Of course, when units are attached to numbers, the units must be taken into account in deciding on the units to be attached to the results of computations. The units of a product or a quotient are just the product or quotient of the units. The units of a sum are the weighted average of the units, which makes it convenient to add terms only when they have the same units. There is no reason at all why numbers may not appear with such units as dollars squared, tons of coal squared, or time per unit dollar. indeed such units do appear in modern economic theory. Nor can I see any objection to fractions involving mixed units except that they may be inconvenient.

Kenneth O. May
Carleton College

Dear Sir,

Far be it from me to take issue with the likes of Augustus De Morgan, or for that matter with your editor who italicized parts of De Morgan's quotation on page 25 of January's The Mathematics Teacher, but . . .

Perhaps for saying this my head too should roll with that of the unnamed fellow of Cambridge on the pretext of being "too severe an examiner," but it seems to me that De Morgan was acting a bit high-handed about the whole thing. Dipping into my own personal muddle of first principles I seem to come up with the notion that division is the inverse of that binary operation multiplication, such that $ax=b$. By the commutative law this expression could have equally well been written $xa=b$, and so I fail to see why the x must of necessity be the multiplicand and not the multiplier. It seems to me that it could be either. So long as division is defined in this way—a process for determining the missing factor in a multiplication—I see no reason why this missing factor could not be the multiplier—the abstract value, the number which says how many times the multiplicand is repeated. After all, this notion is at the very heart of our concept of measurement. What principle is violated when one determines how many $2 meals he can get for a ten-dollar bill by dividing $10 by $2? If De Morgan is right then it seems to me that an awful lot of textbook writers as well as shallow numerists like myself had better fall back and regroup.

Dubiously yours
Francis J. Mueller
Department of Mathematics
State Teachers College
Towson, Maryland

Editor's Note: The Editor should have been more explicit. The item was included to show how mathematics changes. De Morgan is outdated.

Dear Editor,

. . . I wish to comment on Mr. Yeshurun's article entitled "Let's Guess it First." I believe the general idea of getting a clue to a solution from a check of a guess has immense possibilities for the students for whom the framing of equations for problems seems especially difficult, and I have been exploring some of these possibilities in my dissertation. I am surprised that Nyberg's version of this plan, as illustrated in the *Seventh Yearbook* of the Council, has not been mentioned by way of comparison. It may also be pointed out that Ahmes would have worked Mr. Yeshurun's first problem (which sounds like an Egyptian problem, anyhow) in a very similar manner, and historians of mathematics refer to it as an application of the "Rule of False Position." Nyberg acknowledges this similarity in his article.

I haven't had the opportunity to more than glance at the rest of this issue, but I want you to know that I am enjoying the issues regularly, and feel that the articles are a fine contribution to educational literature.

Very truly yours,
Herbert F. Miller
Northern Illinois State Teachers College
De Kalb, Illinois

Reviews and evaluations

Edited by Richard D. Crumley, University of South Carolina, Columbia, South Carolina and Roderick C. McLennan, Arlington Heights High School, Arlington Heights, Illinois

BOOKS

An Analytical Calculus for School and University (Volume I), E. A. Maxwell, New York, Cambridge University Press, 1954. Cloth, ix+165 pp., $2.75.

An Analytical Calculus for School and University (Volume II), E. A. Maxwell, New York, Cambridge University Press, 1954. Cloth, vii+272 pp., $3.50.

The two volumes present in a clear, concise fashion the fundamental processes of differential and integral calculus. A reader from the United States notices the sparing use of italics, the lack of heavy type and boxes to set off formulas, and the small number of exercises at the end of the sections.

The books are written for upper high school and beginning college students. The purpose is to present ideas with a rigor sufficient to the mathematical maturity of the students. Where it is impossible for the author to be mathematically rigorous at the level of his audience he does not hesitate to postulate results or to depend upon descriptive statements to make clear to a student a process. This reviewer was struck by the author's ability to make intuitively clear a knotty point in calculus and yet to write in such a way that no mathematician could legitimately criticize him.

There are no pictures in the two volumes. Where it is absolutely impossible in the opinion of the author to make a clear presentation otherwise a line drawing is used. These drawings are of the classroom variety, the sort a good teacher would use in working with a group. The appearance of the pages is pleasing. The books do not pretend to do anything else than to present those concepts of differential and integral calculus necessary for a study of analysis or engineering applications.—*Myron F. Rosskopf, Teachers College, Columbia University, New York.*

The Compleat Strategyst, being a primer on the theory of games of strategy, J. D. Williams, New York, McGraw-Hill Book Co., 1954. Cloth, xiii+234 pp., $4.75.

This book is written for the purpose of acquainting the general public with some of the basic ideas of the theory of games. This mathematical field, which is quite new and appears to be of growing importance, has for the most part been the domain of a comparatively small group of highly trained mathematicians. By avoiding the symbolism of mathematics and the highly technical aspects of the subject, the author has given an elementary explanation of the fundamental notions of the theory. Although the underlying principles involve a considerable amount of advanced mathematics, one needs only a knowledge of arithmetic (including negative numbers) and the willingness to expend some intellectual effort, to profitably read the book. The style is light and humorous. Many examples are given, most of them rather facetious, but underneath it all one sees how the methodology might well be applied to more serious situations involving conflicting interests. Here is a good chance for students and others interested in secondary or undergraduate college mathematics to get a glimpse of one of the more recent applications of mathematics.—*Fred W. Lott, Iowa State Teachers College, Cedar Falls, Iowa.*

Elements of Algebra, Howard Levi, New York, Chelsea Publishing Company, 1954. Cloth, 160 pp., $3.25.

Instead of presenting the reader with a set of techniques for the manipulation of algebraic symbols as do most books with a similar title, *Elements of Algebra* is concerned with the construction and structure of number systems.

The set of cardinal numbers is defined in terms of the fundamental notion of a set. After the operations in the system are defined, the usual commutative, associative and distributive properties are proved from the basic definitions. The structure of the cardinal number system is discussed from an abstract point of view, followed by general requirements imposed on any set to be a number system. An integer is defined to be a set of ordered pairs of cardinals. Properties of the set of integers are then derived as the logical consequences of the definitions in the system. This general pattern is continued through the book with the rational and real number systems. The appendix contains a discussion of cardinal numbers associated with infinite sets, Peano's axioms, mathematical induction and a very brief mention of groups, rings and fields.

The book is well written. Frequently after the statement of a theorem and before its proof, a note is given to explain the significance of the theorem. The proofs are accurate and for the most part complete. There are a few misprints, but they are not serious enough to cause difficulty. Many of the traditional topics of algebra

are omitted, indeed the brief mention of logarithms in the chapter on the structure of the real number system almost seems out of place in a book of this type.

The reader who masters the contents of this book will have had a stimulating intellectual experience, a good basis for understanding the mathematical manipulations of algebra, and an acquaintance with modern mathematical thought.—*Fred W. Lott, Iowa State Teachers College, Cedar Falls, Iowa.*

Fundamentals of College Mathematics, John C. Brixey, Richard V. Andree, New York, Henry Holt and Co., 1954. Cloth vi+609 pp., $5.90.

This book presents a course in which college algebra, analytic geometry (plane and solid), elementary calculus, statistics, and trigonometry are interwoven. It might serve particularly well for a terminal course in mathematics for college freshmen who do not plan to study more mathematics, although it can also give a good foundation for those continuing in mathematics. By suggested selections of chapters the book can be adapted to various purposes.

The point is made that the book is designed to be read by the student. Right from the start the explanations are careful and detailed, sound mathematically, and yet seeming to speak to the student. The exercises are interesting, sometimes amusing (involving characters such as Calvin Butterball), with many applications to engineering, chemistry, and physics. There are interesting and stimulating references to later mathematics. And yet as one reads on, it seems too often as though the student would be discouraged and perhaps lost because important proofs (such as the proof of the Remainder Theorem and development of Synthetic Division and of $d(u^n)/dx$ are given by a method which seems more of a display of mathematical ingenuity than a direct, meaningful approach. (On the other hand, the use of analytic geometry methods in proving the law of cosines and cos (A−B) seems good.) There is a tendency sometimes to use symbolism which is more complex than necessary, and therefore confusing. In the trigonometry the distinction between trigonometric functions of *angles* and of *numbers* seems awkward and unnecessary. And one feels that in an effort to cover a lot of ground quickly, some sections lose clarity by their sketchiness and by a mere statement of facts or formulas without enough discussion to give them meaning.

There are many good features of the book. There is a remarkable chapter on mathematical statistics, perhaps overambitious, but a good effort at recognizing the growing importance of statistics. The last chapter gives brief discussions and references on a number of topics such as matrices, modular systems, Moebius strips, perfect numbers, and the four color problem. These should be very stimulating. Slide rule is introduced as a special type of nomogram.

Throughout the work on algebra and analytic geometry, many of the illustrations are chosen so as to sow seeds of the ideas and skills needed for the calculus. This interweaving of the subject matter is well done. In connection with using approximate numbers there is much good emphasis on estimating results and using scientific notation, but there is no discussion of rounding off approximate answers to a reasonable number of significant digits.

There is a good reading list. The four-place tables are clear and easy to read, and there is a separate card with tables to use in tests. The answers to the odd numbered problems are particularly complete, including skeleton solutions and graphs. The introduction includes some hints on how to study.

The book is very handsomely bound. The pages are attractive in appearance although the explanatory material and the examples tend to run together on the page. The margins are particularly wide and the student is encouraged to use them for writing notes.—*Henry Swain, New Trier Township High School, Winnetka, Illinois.*

The Geometry of René Descartes, translated from the French and Latin by David Eugene Smith and Marcia L. Latham, New York, Dover Publications, 1954. Cloth, vi+243 pp., $2.95; paper, $1.50.

This is not a textbook in analytic geometry nor a popular exposition of the methods of analytic geometry. It is instead a glimpse into the mathematics, philosophy, and logic of the early seventeenth century, a glimpse afforded by the writings of the man acclaimed by many historians as the father of analytic geometry.

The subject matter of *Geometry* is devoted to solutions of certain classical construction problems which had baffled mathematicians before Descartes' time. However, it was the insight into Descartes' method rather than the problems he solved which impressed this reviewer. You can forget your knowledge of analytic geometry and be led to discover its beauty anew as you follow Descartes: "First, I suppose the thing done, and since so many lines are confusing, I may simplify matters by considering one of the given lines and one of those to be drawn . . . as the principal lines, to which I shall try to refer all others. Call the segments of the line AB between A and B, x, and call BC, y" (p. 29). It may come as a surprise to some readers that, in this first use of the methods of analytic geometry, Descartes did not use mutually perpendicular lines for coordinates axes. In fact, he did not use the term coordinate. Perhaps the closest he comes to a statement characterizing analytic geometry as we think of it today is the following; " . . . all points of those curves . . . which admit of precise and exact measurement must bear a definite relation to all points of a straight line, and this relation must be expressed by means of a single equation" (p. 48).

Although of geometric origin, the problems of this book are solved by almost purely algebraic means. In Book III, the development begins to resemble an exposition of the theory of equations. Some of the topics are: "How many roots each equation can have," "What are false roots?" (negative roots), reducing the degree of equations, and trisecting an angle (using a parabola). In this section you can get a thrill seeing Descartes' "Rule of Signs" as it was originally presented.

To the person who enjoys renewing his abilities in reading French, this edition has a facsimile page of the original 1637 edition facing each page of the translation.—*Lyman C. Peck, Iowa State Teachers College, Cedar Falls, Iowa.*

Introduction to Modern Algebra and Matrix Theory, Ross A. Beaumont and Richard W. Ball, New York, Rinehart and Co., Inc., 1954. Cloth, v+331 pp., $6.00.

The authors have presented us with an introduction to modern algebra which can be read in large sections at a time rather than one which must be painstakingly unraveled phrase by phrase. There is a continuity and a unity to this volume which is to be commended. One topic follows almost effortlessly from another and there is a wealth of illustrative material to reinforce the new ideas developed in the text.

After a brief discussion of positive integers, the algebra of matrices is developed. At first the matric elements are complex numbers but eventually polynomial elements are considered. Groups of matrix transformations and the corresponding coordinate transformations in vector space are also investigated. Enough text and problem material is provided for a one semester course in matrix theory. In the remaining chapters the theory of abstract groups, rings, and fields is developed. The properties of polynomials and the theory of polynomial equations are also treated.

If you wish to know more about modern algebra, this text, tempered in the classroom, is for you. The simple expository style together with the many interesting illustrative examples serves to remind us that learning can be relatively painless even in the more advanced areas.—*A. Schurrer, Iowa State Teachers College, Cedar Falls, Iowa.*

The Japanese Abacus: Its Use and Theory, Takashi Kojima, Tokyo, Japan, and Rutland, New York, Charles E. Tuttle Company, 1954. Paper, 102 pp., $1.00.

Any high school or classroom library will welcome this paper bound book for the pupil who is particularly interested in operating an abacus. The book concerns itself with the Japanese abacus or soroban which has four beads below the bridge and one above. The effectiveness of the book is increased with the addition of an actual soroban, which the publisher is equipped to supply.

The book contains a comparison of the abacus with the electric calculating machine, a survey of the history and development of the abacus, basic principles of abacus calculation, the four fundamental operations with bead manipulation and many exercises to provide experience and rapidity of computation, including time to be allowed for each group.—*Martha Hildebrandt, Proviso Township High School, Maywood, Illinois.*

Numbers: Fun and Facts, J. Newton Friend, New York, Charles Scribner's Sons, 1954. Cloth, xi+208, pp., $2.75.

Dr. Friend has done a marvelous job of writing on a subject that is not often presented in as interesting a manner. *Numbers: Fun and Facts*, was written to emphasize the origin, peculiarities, traditions, legends, and superstitions of numbers. Puzzles and number arrangements of common and infrequent occurrence are presented in an interesting, easily read style.

There are no pictures and a minimum of line drawings, although the quantity of illustrations appears adequate.

Teachers at all educational levels will find topics of value to their students. Sponsors of mathematics clubs may wish to use the book as a basis for increasing the members' knowledge of numbers.—*Roderick C. McLennan.*

Technical Mathematics, Harold S. Rice and Raymond M. Knight, New York, McGraw-Hill Book Company, Inc., 1954. Cloth, xiv+748 pp. $6.50.

This is an introductory text written for students in junior colleges and technical institutes. Enough material is included for a course lasting from one to one and a half years. The organization is such that it would not be difficult for the teacher to adapt the teaching sequence to his own needs. The purpose of the text is to prepare students for employment as engineering technicians equipped to apply fundamental mathematics to the problems of industry.

Arithmetic, methods of computation, intuitive geometry, classical algebra, trigonometry, periodic motion, and brief treatments of analytical geometry and vector algebra are the major areas covered. Included are nearly 5,000 problems which are characteristic of many fields of engineering. Answers to most of the odd-numbered problems are bound in the book, and a separate answer pamphlet is in preparation. Five-place tables are included in the appendix.

The format of this large book is quite good. The exposition, in general, is clear though necessarily brief. A notable exception to the clarity is the definition of "precision" on page 1 where the term "reliable digit" is used though it has not itself been defined.

The reviewer feels that this is an excellent text for the purpose for which it is intended but should not be substituted for the traditional college courses.—*C. H. Lindahl, Iowa State College, Ames, Iowa.*

2222 Review Questions for Surveyors, Russell C. Brinker, Box 153, Blacksburg, Va., 1952. Paper, 169 pp., $3.00.

This is a technical publication intended for surveyors. Every phase of surveying is reviewed. This includes the use of instruments, maps, public lands, and advanced surveying and civil engineering. It will be valuable for use in schools teaching surveying and civil engineering and for surveyors who wish to review all phases of surveying.

It is too technical for use with field work classes in mathematics.—*C. M. Shuster, State Teachers College, Trenton, New Jersey.*

Vector and Tensor Analysis, G. E. Hay, New York, Dover Publications, Inc., 1953. Cloth, v+193 pp., $2.75; paper, $1.50.

The author of this book has, in a minimum of space, managed to present a usable and readable introduction to vector and tensor analysis. Each section of the theory is developed in turn and then numerous and varied applications are made to geometry and physics. Nearly 200 problems ranging from the simple to the moderately difficult accompany the text and are distributed so that the student may employ his newly acquired skills at strategic points along the way. The clarity and brevity of the exposition are sufficient to recommend this volume to the reader who, either as a member of a class or on his own, wishes to become better acquainted with this useful and interesting area of mathematics.—*A. Schurrer, Iowa State Teachers College, Cedar Falls, Iowa.*

BOOKLETS

Moderns Make Money Behave, Educational Division, Institute of Life Insurance, 488 Madison Avenue, New York 22, New York. Booklet, $8\frac{1}{2}'' \times 11''$, 14 pp. free.

This publication is concerned with the subject of money management. More specifically it deals with two concepts relative to sound money management: (1) planned spending and saving and (2) the development of financial security through an adequate life insurance program. The booklet is designed to answer questions concerning the spending of money, saving of money, intelligent planning of the use of money, and the purchase of life insurance. Individuals and their management of money are treated first and then the same principles involving the listing of such basic needs as fixed expenses, flexible expenses, and savings are applied to family money management. Life insurance is described as a method by which families might meet the threat of loss of income through death and loss of income by retirement. The various kinds of policies are listed, and the characteristics and functions of each are briefly described. Throughout the booklet, the emphasis is on tailoring budgets and life insurance programs to fit the needs of individual families.

Very little mathematical material is contained in this booklet. Furthermore, in treating both budgeting and insurance-programing, the booklet undertakes the task of covering a great deal of material in a short space. Despite these limitations the booklet would be very useful for supplementary reading in appropriate mathematics, homemaking, or consumer education courses. The booklet is both well written and well illustrated and very readable at both the junior and senior high school level.—*Clarence Olander, St. Louis Park, Minnesota.*

Using Credit Instruments, Farm Credit Administration, Division of Information and Extension, Washington, D.C. Booklet, $6'' \times 9''$, 31 pp., free.

This booklet discusses the use of credit instruments commonly employed by farmers. Contracts, checks, notes, drafts, bonds, bills of lading, deeds, mortgages, leases, and assignments of income are some of those included in this publication. Each credit instrument is first defined and then a discussion follows which usually indicates the legal consequences and effects of the instruments together with suggestions on how to avoid the more common errors in connection with their use.

This booklet is well prepared and well written in a concise form. It includes little mathematical material but could be used for supplementary reading for advanced students in courses in business arithmetic, practical business course, vocational courses of various types, and courses in agricultural education. It could very well be used for students and others in different groups who would have occasion to use such instruments. To be used most effectively this publication would have to be supplemented by actual forms of the instruments being discussed. —*Clarence Olander, St. Louis Park, Minnesota.*

DEVICE

Radian and Circle Demonstrator (Cat. #7501), W. M. Welch Scientific Company, 1515 Sedgwick Street, Chicago 10, Illinois. 21" disk made of composition-board with cord attached, $7.50.

The purpose of this device, designed by Dr. Harald C. Jensen, is to demonstrate basic concepts of the circle and radian measurement. The disk is made of $\frac{1}{4}''$ hard composition-board, is 21" in diameter, has 60° sectors enameled alternately black and white, has a handle mounted on the back side, and has a 66" cord attached at the edge. The edge has a groove to accommodate the cord which is just long enough to wrap around the disk.

The disk is well constructed and finished. The size of one radian can easily be shown by marking a length equal to the radius on the cord and by then wrapping the cord partially around

the disk. Since the cord can be demonstrated to be equal in length to the circumference of the "circle," the ratio of circumference to diameter can be shown by wrapping the cord around the diameter of the disk. This device can be used effectively at several grade levels, so it might well be located in the central instructional materials center of a school system.—*Richard D. Crumley.*

EQUIPMENT

Logarithm and Trigonometric Functions Chart (*Cat. #7550*). W. M. Welch Scientific Company, 1515 Sedgwick Street, Chicago 10, Illinois. Reversible chart, 76″×52″, roller and hardware included, $15.00.

Two charts, one appearing on each side of heavy white chart stock, present the entire four-place table of logarithms and a four-place table of trigonometric functions for each degree and fifths of a degree from 0° to 90°. The numbers are ¾″ high, and a horizontal black line separates each group of five rows. Each five-row group starts with a row of black numbers and alternates with rows of red numbers. Aluminum strips are attached at the top and bottom of the chart. The chart is accompanied by a hollow, cardboard tube (finished in black), brass hardware, and cord—all of which serve to mount on a wall a roller arrangement for reversing the chart. If preferred, the chart can be hung from two nails.

This chart has excellent quality, and the roller arrangement for reversing the chart is practically foolproof. The numbers can be read from a distance of thirty feet by persons of normal vision, so every student in an ordinary size classroom should be able to see the chart clearly. The chart has a protective coating which makes it washable. The chart will be found to be very useful to teachers in explaining the use of tables of the trigonometric functions and tables of logarithms.—*Richard D. Crumley.*

Educational arithmetic

Ronald B. Thompson, of The Ohio State University, has prepared "A Supplement to College Age Population Trends, 1945–1970" for the American Association of Collegiate Registrars and Admissions Officers. Some of the figures he has compiled tell a story without words about the predicament of the elementary schools and the impending problem in the high schools and colleges.

Number of Children

Year	*Age 6–11*	*Age 12–17*	*Age 18–21*
1933	13,704,709		
1939	12,713,765	13,653,324	8,455,935
1943	12,205,137	13,173,495	9,060,592
1949	14,317,912	12,138,288	8,584,336
1954	17,921,998	13,552,629	7,967,556
1959	20,891,600	17,142,295	8,785,930
1964		20,494,660	10,955,207
1970			13,609,830

The years 1943, 1949, and 1954 recorded progressive low points in the elementary school, high school, and college age populations, respectively. In 1949, the depression drop was wiped out in the age 6–11 group; in 1954, there were 4,000,000 more youngsters of elementary school age than in 1933, when the decline set in. In 1959, there will be over 7,000,000 more. In the age 12–17 group, the turn came in 1949, the depression deficit was wiped out this year, and by 1956 our schools must be prepared to handle 7,000,000 more than at present. The age 18–21 group will begin its upward climb in 1955, correct depression losses in 1958, pass the 1944 peak of 9,087,788 in 1960, and move some 4,500,000 beyond it by 1970. Obviously, our gross educational product must occupy as much of our thought and attention as our gross national product for some years to come.—*Newsletter #72*, January 21, 1955, Engineering Manpower Commission of Engineers Joint Council.

Teacher: "Tommy, if you had 20 sheep in a field, and 5 got out, how many would there be in the field?"
"Not any."
Teacher: "Tommy, you don't know your arithmetic."
Tommy: "Teacher, you don't know your sheep!"

• **TIPS FOR BEGINNERS**

Edited by Dr. Francis G. Lankford, Jr., Department of Education, University of Virginia, Charlottesville, Virginia

Better bulletin boards

by Humphrey C. Jackson, Teacher, Parcells Junior High School, Grosse Pointe Woods, Michigan

Many teachers of mathematics have found the bulletin board a helpful aid in teaching, and a challenge to their ingenuity and that of their pupils. From my experience in using bulletin boards, I have accumulated some ideas which I should like to share with other teachers in the paragraphs that follow.

When planning a bulletin board display, the first thought should be directed to the *theme* of the display. For instance, if the class is studying per cents the bulletin board should carry out an idea definitely related to this topic. *Frequent change* of material is also necessary to keep pupil interest alive.

Stress should be placed on *simplicity*. A few ideas presented clearly is better than a conglomerate display of many ideas. A simple display tells its own story briefly and effectively. For instance, when arranging a display related to investments, I have cut pieces of ticker tape to form the letters of the title of the display.

There should be no doubt about the purpose of the display. I have arranged a number of graphs made by pupils in such a way as to spell the word "graph." Frequently pupils will find a picture which may be used to stimulate thinking along a particular topic. Silhouettes and the spelling out of words in colored construction paper against a background which contrasts harmoniously is very effective.

Be sure to have a carefully worked out *color scheme*. It is usually unwise to use many colors in one display. Best color arrangements can be obtained by the use of the complementary colors. One possibility is to select colors in harmony with the season. The fall months suggest browns, reds, oranges and yellows; December suggests reds and greens; January snow suggests white and blue. Red, white, and blue are appropriate for February: green and white for March; and the spring months suggest pastel colors. Whatever the display may be, use color effectively, and sometimes for relief use just black and white.

Here are some materials I have used or seen other teachers use. Thumb tacks may be used to spell out words. You may get the effect of a third dimension by using letters fastened by pins which project out about three-fourths of an inch from the bulletin board. The paper letters are kept away from the board close to the heads of the pins. Paper folding can be used to get sculptured effects. Several references which I have found useful are listed at the end of this paper.

Left-over rolls of wall paper which pupils will bring to school can be used effectively as background for a bulletin board display. If the right side of the paper is too colorful, the reverse side may be used. I have also used colored chalk as well as bright colored construction paper. There is on the market paper made especially for display purposes which has a waffle design pressed into the paper, or is corrugated or velvet textured. This paper comes in assorted widths up to 60 inches and in

rolls 25 feet or more in length. It is usually sold by window display firms or artists' supply stores.

If your school is fortunate enough to own a *pantograph*, you will be able to enlarge outline types of materials for your bulletin board. If you do not have a pantograph available, it is a simple job to construct one. Most libraries have descriptions of this drawing instrument. Perhaps you could arrange for the shop teacher to help one of your students make a pantograph for you.

Quotations and proverbs such as "God eternally geometrizes" (Plato) or "Mathematics is the Queen of the Sciences", have a place on a bulletin board. I have used a new quotation for each week of the school year. We are sometimes unaware of the impressions these quotations make upon our pupils. I was happily surprised to learn from a former pupil who visited me that she had copied and kept every quotation which had been displayed on our bulletin board.

There are many free materials available for bulletin board use. Information on many of these appeared in the department of THE MATHEMATICS TEACHER, "Aids to Teaching," during the years 1950 to 1954.

It is possible to purchase letter patterns which can be traced on construction paper and cut out. These are made of a heavy type tag-board and will last for a long time. Variations in style of letters may be obtained by notching the letters at irregular intervals, or if you desire a modern type of letter, try leaving solid the hollow loops in letters like B, R, P.

Cartoons and jokes can be used effectively, especially when they pertain to the subject under study. One teacher I know has a "Just for Fun" portion on her bulletin board which encourages pupils to bring in such material for display.

Frequent displays of pupils' work can be exceedingly successful. It is helpful to appoint pupil committees to plan and carry out such displays, especially for P. T. A. meetings. Parents are always happy to see examples of the good work their children have done.

You will find that symmetrical arrangements are easier to achieve successfully. It is also true that when several illustrations or papers of the same size are to be displayed, it will be more attractive to vary the arrangement. For example, do not produce a monotonous effect by merely placing papers in several rows. Instead, try to arrange such material in groups or in alternate positions.

Good bulletin board displays will inspire and encourage pupils in their study of mathematics. Often displays will stimulate questions in the minds of pupils which may be the basis of a good class discussion. Finally, when you and your pupils have arranged a fine display, it is good to take a kodachrome picture of it. This can be used as a challenge to other classes, and possibly for publication in the school newspaper.

SUGGESTED REFERENCES

MARY GRACE JOHNSTON, *Paper Sculpture*. Worcester 8, Massachusetts: The David Press. Inc.

TADEUSZLIPSKI, *Paper Sculpture*, New York: The Studio.

SAMUEL I. JONES, *Mathematical Wrinkles*, S. I. Jones Co., 1122 Belvidere Drive, Nashville 4, Tennessee.

BURTON E. STEVENSON, *The Home Book of Quotations*, New York: Dodd, Mead and Co.

New quantity discounts

In order to keep their practices in line with those of other publishers, the NEA and its departments are revising their schedule of quantity discounts. The new schedule will not cause any change on quantities of from one through nine copies. On orders for 10 copies or more, new discounts will apply.

Beginning with June 1, 1955, Council publications will be sold according to the new discount schedule. On quantity lots of the same item, the discount will be as follows: 2–9 copies, 10%; 10 or more copies, 20%. Bookstores and agencies that buy publications for resale purposes will be allowed a straight 20% discount on any quantity. Orders will be shipped postage prepaid only if remittance accompanies them.

Edited by John A. Brown, University of Wisconsin, Madison 6, Wisconsin and Houston T. Karnes, Louisiana State University, Baton Rouge 3, Louisiana

An experiment in teaching[1]

Contributed by Beatrice Buzzetti, Cassville, Missouri

In the Bremerton school system only those eighth-grade students who receive an arithmetic grade of B or above and those recommended by their mathematics teachers are permitted to study algebra in the ninth grade. Since our senior high-school guidance department urges all students who hope to attend institutions of higher learning to elect beginning algebra sometime during their high-school career, Bremerton gets an older group but a slower group of first-year algebra students.

After three years of blundering and more or less blindly experimenting with these classes I decided to treat the work as a special project, using my past experience as a basis on which to formulate a definite plan for the year's work. The announcement of this course in curriculum by Dr. Edgar Draper of the University of Washington crystallized my thoughts into action for I realized that his advice and encouragement would be of immeasurable value to me. At once I set out on a diligent search for any appropriate teaching aids and it is surprising what one can find even in the field of mathematics. I began writing up everything I said and did in class that I might sift out the best of it in future years and add other material.

Thus far I have attained some measure of success on which I hope to improve after still more time and experience. I believe that the heart of what success I have achieved lies more in the psychology used than in the caliber of teaching done on the subject matter. Students who are slow in arithmetic are not, in general, top students in other subjects. Since it is natural that teachers enjoy the better students, these boys and girls have gone through school life feeling unwanted and inferior and some have developed a complex against teachers and against school. So my first objective was to build up their ego, and to implant in them the belief that to teach them was a challenge which I would enjoy; to make them feel that perhaps they had misunderstood many of their teachers. I tried to bring them to realize that most teachers love their work, else they would not make the sacrifices they do for teaching. I sensed a feeling in the room of surprise, yet one of warm welcome as I read this excerpt from William Lyon Phelps.

> I do not know that I could make entirely clear to an outsider the pleasure I have in teaching. I had rather earn my living by teaching than in any other way. In my mind, teaching is not merely a life-work, a profession, an occupation, a struggle; it is a passion. I love to teach, I love to teach as a painter loves to paint, as a musician loves to play, as a singer loves to sing, as a strong man rejoices to run a race. Teaching is an art—an art so great and so difficult to master that a man or a woman can spend a long life at it, without realizing much more than his limitations and mistakes, and his distance from the ideal . . . .
>
> So far from being dull routine, teaching is to me the most exciting, the most thrilling of professions. It has its perils, its discouragements, its successes, its delights . . . . Whenever I enter a classroom filled with young men, I think of them not as a class or as a group, but as a collection of individual personalities more complex, more delicate, more intricate than any

[1] A report given before the class in curriculum building, February 15, 1951.

machinery. Not only is every student an organism more sensitive than any mechanical product; every student is infinitely precious to some parent or to some relative who may be three thousand miles away. That is why the teacher should never use irony or sarcasm or the language that humiliates; that is why he should never take the attitude of suspicion or depreciation. The officials at the United States Mint, the Head of a diamond mine, the President of a metropolitan bank are not dealing with materials as valuable as that in the hands of the teacher. Their mistakes are not so disastrous as his; their success is not so important. The excitement of teaching comes from the fact that one is teaching a subject one loves to individuals who are worth more than all the money in the world.[2]

This year I spent the first five days talking to my students. In my discussions I talked of many things, pointing out that training in school is primarily to develop good citizens and that because I enjoy mathematics I have chosen this field as my means for this training. I suggested that many are taking mathematics only because their parents want them to or only to attend a certain university—that perhaps they don't like mathematics and don't want to take it; but then I assured them that they could grow through the study of algebra even if they learned nothing of algebra. I assured them that I would make their experience in the course worth while and that if only they would cooperate they would earn a credit. I stated that perhaps some were the victims of suggestions by others that mathematics is hard—that they might surprise themselves once they really tackled the subject with determination. One student who has actually experienced this brought me a poem a few days ago—I shall read it.

It's All in the State of Mind

If you think you are beaten, you are,
If you think you dare not, you don't;
If you like to win, but you think you can't
It's almost certain you won't.
If you think you'll lose, you've lost,
For out in the world you find
Success begins with a fellow's will;
It's all in the state of mind.
Full many a race is lost
Ere ever a step is run;
And many a coward fails;
Ere ever his work's begun.

Think big and your deeds will grow,
Think small and you'll fall behind,
Think that you can and you will;
It's all in the state of mind.
If you think you're outclassed you are;
You've got to be sure of yourself before
You can ever win a prize.
Life's battles don't always go
To the stronger or faster man,
But sooner or later the man who wins
Is the fellow who thinks he can.

I talked of employers and the type of people they wished to hire—on the job every day and on time; and that our habits for life are formed in the public schools. Employers look to the schools for recommendations of student ability and punctuality. At the end of these discussions I had them answer a short questionnaire about past work and future plans in which an important item was to tell me *how* they were going to overcome past difficulties in mathematics. I tried in my discussions to bring out some methods and I hoped that they would ponder over this and perhaps recall these suggestions. This was given as a definite homework assignment—their first. Finally, when many were "chomping at the bit," I issued texts.

Incidentally, this five-day "indoctrinating" period gave any late-comers a chance to feel that they had not missed out—and truly they had not, for when 27 or 28 students in the class have made up their minds to work and to cooperate, the other 2 or 3 are bound to fall in line even though they missed the discussions which brought about the fine spirit of the group.

I began my teaching with detailed explanations and short assignments. I have found that no matter how definite an assignment, it is far better to have one little corner of the blackboard where this is written in detail every day. I proceeded at about half the normal speed—tried never to appear in a hurry—to stay on a type of problem until the majority of the class grasped the method regardless of the number of days required for mastery.

[2] William Lyon Phelps, ***Autobiography With Letters.***

Frequent ten-minute tests kept them alert. I always explained the problem at the end of the test, graded these papers, and returned them the next day, explaining the problem again so that they might see their own errors.

At frequent intervals I brought in discussions concerning past students or those presently in my classes, always with the thought of further reassuring them of their possible latent ability and of my sincere interest in them. I always discussed in class any student's success or growth—thus it became contagious.

I assigned written work almost every day except over holidays. This is homework for some but the brighter ones sometimes complete their work in class as I tried to have supervised study during part of the period. For these better students I urged extra credit work—in fact no A grades were given without it. I expected them to do extra unassigned work in the text or to repeat any work which we have covered.

At once I noted a very fine attitude in all classes. They were prompt with their work every day. In my preliminary discussions I had told them that work missed was their own responsibility—that I would never check with them about it but that it should be done, and that I would be happy to look it over and give them credit for it. I tried to make them feel that I was doing them a favor in looking it over rather than that they were doing me a favor in making up the work. It has evidently worked, as practically every absent student returns with the previously assigned work completed and makes up the remainder of it within a few days with no urging on my part.

Bremerton teachers usually give out poor work slips at the end of six or seven weeks to be signed by the parent and returned. I sent out eight, all of which were signed and returned within two days. At the end of the quarter all grade cards except one in three classes were returned within two days, and that one came in the third day. This is phenomenal—at least in my classes.

At the end of the first quarter I prepared a lengthy questionnaire concerning the work and the methods used—asking them not to sign their names. Besides the questions, I left space for discussion. I urged the students to be free to speak their thoughts because I was anxious to learn more about how to better my teaching—which can only be done when students are contented. I have conducted many such questionnaires in my classes throughout the years and well know that in so doing I'm asking for unkind remarks. I feel that it is well to give unhappy children a chance to "blow off steam." To my great surprise and delight, not one word of discontent appeared. I am going to be bold enough to read a paragraph from several papers.

I blush at the thought of reading their complimentary remarks. You'll say to yourself, "only an egotist or a person with an inferiority complex could do it," but I'm doing it to show you the attitude students take. I often ask students to help me explain something to a child who looks blank while I explain it and some can actually help. Their minds are at the same level. I've told them about taking this course that I might better my teaching. I had to argue with myself a long time to get up the courage to read these but finally rationalized this way, "Well, it matters not what you think of me if only I can give you some points on how students react to humble teachers." I know that many teachers feel that they should be able to answer all questions a student asks else he will think them incapable. In striving to do this they defeat their own purpose by giving incorrect answers. The following excerpts are, of course, the most outstanding ones but all papers carried a similar theme.

"Algebra to me is not just a kind of math. You've made it more or less of a course in character building than math. That certainly makes it different from school work (drudgery). You've rather

personalized algebra so we feel like we are doing it for ourselves instead of you or a report card."

From another: "You have the best approach toward teaching of anyone I have seen. The fact that you've taught for a number of years and are still after new and better ideas makes the class, I think, feel like you really care about them getting it. Most teachers would have sat down to ride the rest of the way a long time ago, knowing it all!"

This is a short one: "I think algebra has taught me so much it would take more than this one paper to write it all."

From one who has learned to think: "Yes, it already makes you slow down and think, rather than just hurry over it to get it done. This has helped in other subjects also. If I don't use it in the future, I'm using it now and I am forming habits for the future so I think it will help."

And another: "I really approve of the way class is conducted. Everyone can say what they want without being afraid of saying the wrong thing."

From one who has tasted of the joy of mental achievement: "In this class I have learned to work and get my work done. I don't think I have *ever* tried to get work done or make up for absence and tried to get extra work done for extra credit. This has affected my other classes because I have been trying in them and am getting good grades."

And this from an appreciative student: "I think that in all the other subjects I have been just learning the basic parts and stopping there. But in here I am being taught a new way and am learning more than I would otherwise. I feel that in my other classes my teachers are doing the least work in teaching that they have to do, but I feel that you are really putting all you've got for the benefit of the kids that wish to learn and some that don't."

I was consumed with curiosity about the paper containing the last paragraph because a few remarks elsewhere in the report gave evidence of an extremely poor student, so I mentioned that I was curious and stated that if the student who used a ball-point pen would give me his or her name, I would appreciate it. I assured them that my attitude toward the paper was complimentary to the student. As the class passed out a little fellow came up and said, "I'm it." Surprised, I answered, "What?" "I used the ball-point pen." He was the last student I would have guessed for truly I had been very hard on him. I also had him in study hall and because he had pestered me so much I had recently had a special chat with him in which I commanded him to quit wearing out the floor coming to my desk—but rather to come once or twice each day and then really think and try to follow my suggestions for the one example on the rest of them. Teen-agers are most fair minded. He had accepted my command in the right spirit.

The slow student is usually the frequently absent student because he doesn't enjoy school. Many days I had not one absence in all three classes so at the end of thirteen weeks I checked our per cent of absence with that of the school and found the ratio three to five in our favor.

I told the students early in the year of my plan to rearrange them and to group them into three ability groups at the beginning of the second semester. I had the courage to discuss this with them because it was such a success last year, my first year to try it. Contrary to adult opinion, the slowest group last year was the happiest about the arrangement. I learned this through unsigned questionnaires. They contained such remarks as "I feel at home now," and, "I feel free to ask questions for I know that other students in this class probably have the same questions in mind."

The second quarter, however, was most trying to me. The gap between their various levels of ability began showing up more and more. I realized that my better students were forming lazy habits and that my poorest ones were bogging down

under the load. I skipped the more difficult parts of the text, planning to return to that work with only the best group. We worried along to the end of the semester doing what work we could in such varied classes and mostly getting better acquainted and better adjusted to school. My plan, which I frequently discussed with them, was to place all A and B students in one group and to at least double the assignments so that they might cover enough of the text to be able to go on with any advanced mathematics they cared to. The C group would return to the beginning of signed numbers and proceed at their own pace. Since these people were only C students the first semester, they could not hope to receive more than a C grade. That would not be fair to them since they would really only cover about one semester of algebra but they would know it thoroughly. Had we progressed at the regular rate they would probably have failed. They accepted this logic without an argument. The poorest group would be made up of my D, F, and S students and likewise they could not expect a higher grade than they had earned before, at most not higher than D, because we would progress only through about one quarter of the year's work but we would get a good review of arithmetic thrown in. This they accepted in good spirit.

In our early registration, only two students insisted on going into a group higher than I had rated them—this I permitted—five of them told me that I overrated them and requested to be placed in a lower group. I complied with their requests.

At the end of the semester I gave the same standardized test which I gave to my classes last year: Cooperative Algebra Test, form R, from Cooperative Test Service, 15 Amsterdam Avenue, New York. And though I felt that my classes this year were as slow or slower than those of last year, our gain on the percentile rating was 12. When you consider that on these tests the percentile score of 50 is the median, 12 is quite an increase.

We are now in our third week of the second semester and all are happy in their new arrangement. I think of them as my good class, my better class, and my best class. In my best class we are doing the usual work, first picking up sections we skipped. I have made it clear to them that to cover the required amount we must plan to study about one hour outside of class each day and I let them help me in deciding how long an assignment I might give. In my better class I am using a few new tools in the teaching of signed numbers. For example, we all made simple slide rules from tag board (a suggestion I found in an advertisement) to clarify addition and subtraction of signed numbers. They were all interested and many felt that they could grasp the topic much more clearly. Then for practice in addition of signed numbers, we paired off in two and played a game. They loved it and my apprehension of a noisy room came to naught.

I have found many helpful suggestions in THE MATHEMATICS TEACHER, the monthly publication of The National Council of Teachers of Mathematics, and some interesting problems in some of the latest mathematics books for the layman. A catchy problem now and then stimulates interest.

In my good class we are using *Refresher Arithmetic* by Stein. At the beginning of this text there are three inventory tests of 43 questions each. Questions of identical numbers are almost identical and following this there are 43 exercises designed to clear up the difficulties with each of the 43 questions, the theory being that students who answer, say, question number 3 correctly in all tests do not need to study exercise 3. Part II of his book contains material on weights, measure and other practical work.

I have made a chart of all the students and their answers on the 43 questions. Each day I assign the entire group something in part II and all those who missed, say, number 1, move to the front of the room and we work on exercise 1. I try to complete this in one day. The next day,

those who missed number 11 will have that extra work too. Now I realize that this method has its drawbacks—especially in that the student who missed the most will have many large assignments, but I'm hoping that this will work itself out somehow.

I expect to take up the basic elements of algebra during the last quarter. I hope that with this improved foundation they will be able to get the rudiments of algebra.

I am simply trying to make the best of a bad situation. I do not feel that all students who hope to study at institutions of higher learning should be herded into algebra. I feel that they should have individual guidance rather than mass guidance and perhaps this illustration will help to clarify my stand.

Animal School

Author unknown

Once upon a time the animals decided they must do something to meet the problems of the "New World," so they organized a school. They adopted an activity curriculum consisting of running, climbing, swimming and flying and to make it easier to administer, all animals took all subjects.

The duck was excellent in swimming, better in fact, than his instructor, and made passing grades in flying, but he was poor in running. He had to stay after school and also drop swimming to practice running. This was kept up until his web feet were badly worn and he was only average in swimming.

The rabbit started at the top of the class in running, but he had a nervous breakdown from so much make-up work in swimming.

The squirrel was excellent in climbing, until he developed frustration in the flying class, where his teacher made him start from the ground up instead of from tree-top down. He also developed charlie-horse from over-exertion and then got a "C" in climbing and a "D" in running.

At the end of the year, an abnormal eel, who could swim exceedingly well, and also run, climb and fly a little, had the highest average and was valedictorian.

The prairie dogs stayed out of school and fought the tax levy because the administration would not add digging and burrowing to the curriculum. They apprenticed their children to a badger and later joined the groundhogs and gophers to start a successful private school.

In conclusion I want to urge all teachers of these slow groups and perhaps of the better students that time spent in motivation "pays off." I feel that I, too, have grown through this experience. The students' genuine appreciation of my interest in their welfare has been a great joy to me. It has made my life richer and fuller.

A comparison of two methods of teaching a unit on insurance

by Clarence E. Olander, St. Louis Park High School, St. Louis Park, Minnesota

Every teacher is interested in using a method of teaching that will provide for the students the most worth-while and life-like experiences which will lead to final achievement of the desired outcomes. Most of us have read a great deal of literature praising the merits of the unit approach as a method of teaching. Some of us have become enthusiastic about the use of this method and probably have employed it successfully in our classes. Despite all the acclaim accorded the use of units of instruction, very little research has been conducted to find out how effective the unit approach has been in attaining our objectives. Thus, it would seem worth while to investigate the effectiveness of the unit approach in teaching a topic in mathematics.

This article describes an experiment designed to test the relative effectiveness of the unit approach in teaching the topic of insurance. Insurance was selected as the experimental topic in this study because it lends itself to presentation by both the unit and traditional methods of teaching. In addition, most mathematics textbooks at the eighth-grade level include

some content involving insurance. Insurance also offers possibilities concerning the social aspects of mathematics as well as the computational and problem-solving aspects. Two eighth-grade mathematics classes were selected to take part in this study. The two classes were not grouped according to ability of any kind. They represent an administrative selection based upon class schedule similarities. Rigorous selection techniques for obtaining a random sample were not employed. One of the classes was randomly selected to be the experimental group while the other served as the control group. The experimental factor was the use of the unit approach. This was present in the experimental group while in the control group the traditional method of presentation was employed. Variables such as subject-matter content and teaching time were controlled by requiring them to be the same in the experimental and control classes. The variable under investigation was the method of presentation.

As previously mentioned, the experimental group used the unit plan of instruction. The writer constructed a unit on insurance which served as the basis for presenting the material to the experimental group. The unit approach included many student activities and a great variety of materials of instruction. Life insurance was selected as the type of insurance to illustrate the basic concepts of all insurance primarily because of the large amount of material on this topic available in classroom quantities. Other kinds of insurance such as fire, auto, hospitalization, and social were studied through the use of committees. Each committee studied a particular kind of insurance and then presented a group report to the class. The committees engaged in various activities during the preparation of their reports. The activities included the construction of scrapbooks on a particular kind of insurance, visits to insurance companies, talks with insurance agents, and the collection of material from magazines, newspapers, and the library. During the course of the unit each student engaged in some project relevant to the study of insurance. Some examples of the projects carried out by students were debates on different types of insurance, exhibits of insurance materials including posters, construction of graphs and charts showing various factual information concerning insurance, and special reports on the history of insurance. In the experimental group problems relating to both the mathematical and social aspects of insurance were solved and discussed.

The control group was presented the same content for the same length of time through the use of the traditional method of teaching. A standard eighth-grade mathematics textbook served as the basic source of information for the control group. This was supplemented by occasional lectures from the teacher. A typical period for the control group was about equally divided between working problems and discussing the problems together with their social applications. The student activities were restricted to the working of problems relating to the different types of insurance.

At the beginning of the study each group was given a basic skills test on arithmetic, a pretest on insurance, and a mental ability test. These tests were of importance in determining the skills, abilities, needs, interests, and readiness of the students for the topic to be studied. They were used to hold constant the effects of mental ability and initial information. The pretest was again administered at the end of the study to measure the learning of the students in regards to the desired outcomes. The scores of the students on these four tests were used as the basis for the statistical analysis. The significance of the differences between the means of the experimental and control classes was tested by Fisher's "t" test. The experimenter used the F distribution to test the assumption of equal variances. More adequate and powerful techniques for this situation such as the methods of analysis of variance and covariance are available but were not used in

this situation. However, much of the basic data necessary for these techniques has been computed and could be used in conducting further research on this problem. The writer is well aware of the limitations of the use of the "t" test in this situation. However this fact is taken into consideration in stating the conclusions.

For each of the four tests the null hypothesis that the two populations have the same variance was accepted. It was necessary to show this since in using the "t" test the assumption is made that the variances of the two populations are equal. Similarly, in each case the null hypothesis stating that the two populations have the same mean was accepted. In other words, the means of the two classes for each of the four tests involved are not significantly different. The pedagogical conclusion is that there appears to be no real difference between the average achievement of the two classes as measured by the tests used. As previously mentioned, no attempt was made at matching students or rigorously holding constant some of the variables. But on the basis of the "t" test applied to the data of the final achievement it appears that there is no significant difference in achievement, regardless of the method used to present the subject matter. The experimental group attained a greater mean of the final test but this difference was not significant. It may have been due to the higher mean scores accomplished by this group on the pretest, basic skills test, and intelligence test.

Certain implications for teaching and further research appear warranted as a result of this study. Although the unit approach did not show up as the superior method, subjective appraisal of the two methods involved would influence the writer to choose the unit approach of presentation in this particular situation. This appraisal is based upon very careful observation of the daily activities of the students in both groups. In the control group using the traditional approach little enthusiasm or interest in the subject was noticeable. However, the group studying under the unit approach expressed a great deal of interest in the subject. This was illustrated by the fact that students were continually bringing in material from home, engaging in extra credit projects, questioning their parents about insurance, and continually doing things to convince the writer that the students enjoyed what they were doing and were interested in the many activities they engaged in. The students reacted very enthusiastically to the opportunity given them to work in committees. The manner in which they cooperated with other students in the preparation of their reports was encouraging to the experimenter. The students verbally expressed their liking of the techniques used in the unit approach and stated that they would enjoy the use of these techniques in studying other topics.

This writer feels that many of the outstanding features of the use of the unit approach of teaching cannot be measured by the use of an objective test such as that used in this experiment. Attempts should also be made to measure the effectiveness of the unit approach in the development of appreciations, attitudes, and insights. The writer did use an attitude test at the close of the study. Fattu's Nomograph was used to analyze the results of the test. The analysis pointed out a significant difference in percentages for several of the items. After studying the results it appears that the attitudes formed in the two groups are not the same. The writer feels that those students studying under the unit approach were exposed to a more varied outlook concerning a particular attitude, and therefore were forced to think more critically about the attitude in question. In further research concerning the unit approach more careful attention might be devoted to measuring the reactions of students to various classroom situations in addition to the use of achievement tests.

Testing instruments played an important role in the outcome of this study. The designing of an instrument which will

measure all the factors upon which an analysis of superiority of method can be made is extremely difficult. In conducting further research more and better instruments to measure all the factors of the desirable outcomes of a teaching situation are the problem of the investigator.

In addition to constructing better tests one would want to use more powerful techniques such as the analysis of variance and covariance. It would allow the investigator to draw more general inferences from his data. Repetition would be necessary if the conclusions are to be applied to typical classroom situations. It is only through continued experimentation that continued improvement in method can be accomplished.

In the opinion of the writer, the additional skills such as the ability to locate material and the ability to work cooperatively in committees acquired by the use of a unit approach justify the use of such an approach in some situations. The unit plan should probably be used periodically throughout the year whenever a topic is to be presented that is definitely adaptable to employment of the various activities embodied in the unit plan of presentation.

Divide and conquer

by Irwin N. Sokol, Kennard Junior High School, Cleveland, Ohio

This is a report of an experiment in small group teaching performed at Rawlings Junior High School in Cleveland, Ohio. It explains how I solved my particular problem. Perhaps you, too, have had similar frustrations and have not known exactly what to do. This report traces what happened to a class in 7B-5 mathematics during the first semester of the 1952–53 school year.

The 7B-5's were the lowest of our "regular" 7B sections. Twenty-eight members of that class faced me the first day we met. Within two weeks the number had dwindled to 24. (Two were moved up to higher sections; one was sent back to elementary school; and the other transferred to another school.) I received two new enrollees to our school in their place, giving a total of 26 pupils in all.

Since the 7B is usually made up of students graduated from many different elementary schools, I usually test these classes to see how much they know before we begin our journey together. On the Stanford Achievement Test their grade equivalents ranged from 3.7 to 7.5, which is a rather wide range on the lower level. Our sections are supposed to be fairly homogeneous, but they don't always turn out that way. I had a difficult situation before me.

At first I tried the easier, more common, teacher-talk, pupil-respond method, but without much success. I was not reaching enough of my pupils to satisfy either the class or myself. This method was therefore abandoned and I was forced to find another way to teach them.

It took me five weeks after the semester had begun to reach this decision. (Seven of the 26 students did not as yet know their multiplication tables up to nine!)

For a number of years I had thought about some day discarding the recitation, whole-class, teacher-centered method of teaching. In order to replace something, one should have in mind a workable substitute. I had taught a class of 42 eighth-graders on an individualized basis three years ago. The amount of time needed in preparing for each day's lesson was of such great length that I could not see attempting that program again. It did work,

but so did I! The individualized work method did not seem to be the answer.

Perhaps I could adapt it somehow. Teachers of other subjects had used a small-group approach to teaching successfully, but very little had ever been done or at least published on this method and its application to the teaching of mathematics. Maybe it was worth a try. If it didn't succeed, the class would not suffer much more than they already had, and I believed we both could learn something from failure, too.

In the beginning, I gave the whole class a series of short, fifteen-minute small-unit tests on the fundamentals of mathematics. These I had used with my regular sections before, having constructed a battery of thirty of them a few summers before (two forms of fifteen different tests). They covered: (Test 1) understanding and meaning of whole numbers; (Tests 2 to 4) each of the fundamental processes of whole numbers (addition, subtraction, multiplication and division); (Test 5) meaning and understanding of common fractions; (Tests 6 to 10) each of the fundamentals of common fractions; (Test 11) meaning and understanding of decimal fractions; and (Tests 12 to 15) fundamentals of decimal fractions.

Each student was given these tests one at a time until he failed to achieve a satisfactory score on one of them. That point of failure determined where he or she would begin the slow process of arithmetical learning. After checking and sorting all the papers, I divided the class into five small groups, ranging in size from three to seven students in each.

Each day before the class entered the room, I posted on the blackboard a list of the classwork and homework problems I expected them to do. They used the regular seventh-grade textbook. The preparation for this work took from four to six hours' time during my weekends. Naturally, the assignments had to be made very flexible—absences, assemblies, slower learning than I had hoped, etc., took their toll.

For one week I rushed from group to group, helping each with its particular problems. After running myself almost ragged, I finally decided that there must be an easier way. I went to the study-hall and began exploring the myriad of faces for some student helpers whom I knew and who might be dependable. As I had suspected, all the good help had long ago been picked out. I could have the left-overs. I struck gold, though. There were two girls who had somehow remained undiscovered. They were new to our building but they were in the top section of the 8B. They agreed to help me.

Each weekend I prepared for them a list of what they had to do with their groups. This included the classwork assignment, the previous day's homework assignment to be checked, and the new homework assignment, together with samples of how I wanted these problems worked, and where I thought an explanation would be necessary and helpful. Answer books also were provided for their use.

The class was prepared for their new teachers by a few well-chosen words on why we did what we were doing. They were told that they would probably not be able to pass at all if they tried to keep up with each other, and their grades the first few weeks bore that statement out. They decided that cooperation was the only way they could learn—and, incidentally, pass.

My girls were good workers. The class accepted them well. Of course, there was some teasing, but proper actions stopped any future disturbances. It was surprising to me to see how the girls taught their groups. I left them to their own devices at first and stepped in only when I thought I could aid them. Privately, I gave them suggestions to use. Most effective was the use of the blackboard. The group was brought up to the board; they were shown how to work the problems; and then each member was allowed to try out a few problems until understanding developed.

Since there were usually more than three groups operating at one time, and

there were only three "teachers," the class period was divided into two parts. (Our periods were of 45 minutes duration.) Half the class worked on the day's classwork or looked over their homework when they had finished, while the other half worked with the teachers. When this period was over, we left the group with which we were working and went over to the next group assigned to us. If we finished early, we would circulate around the room helping those who seemed to need it, or those who asked for aid. Any student was free to ask for help at any time from his group teacher or from me.

The progress of the class was measured by the aforementioned unit tests. The students could move ahead, remain in the same group, or drop behind, depending upon how well they did on these tests, and how quickly the rest of the groups moved along. There was, therefore, some element of competition to keep up interest. Shifting from one group to the next took place about once a week. Units were planned to end at approximately the same time, although that was not always possible. With the two forms of the test available, I could switch from one to the other if a student failed the test the first time and tried again. They were never given back their test papers, but they knew how well they were progressing by their relative group movement. If there was any question as to why they were placed where they were, they were then shown their papers. (I had not a single complaint all semester—a few disappointments, but no anger.)

Usually the groups varied in number from four to eight, with the mean being five. They seemed to divide that way for no particular or planned reason. Any number more than six caused a problem because there were only three of us. When that occurred, due to a widening of the range of accomplishment, I took as many as four groups myself.

Out of the 26 who remained in the class until the end of the semester, 16 finished the requirements set up by our mathematics department for the class two whole sections higher! (We do not require the same amount of material to be covered in the same amount of time in our lower sections.) Four reached the point that they were supposed to reach had they been taught as a whole complete class. There were six who did not respond as well as I had hoped, but they would not have even attempted the work under any other method. I was able to salvage 20 out of 26. Under the teacher-recitation method, I doubt if I would have had six who could have completed the work of the twenty who went on to the 7A.

My conclusions may be summarized as follows: (1) The class generally worked hard, although there will always be a few who do not respond properly all the time. (2) They did homework without complaint because they knew how to do their work. They had been shown how and they had demonstrated to their teacher that they could do the work—on the board, on their homework papers, and on their tests. Success helps fertilize this type of atmosphere. (3) They knew exactly what to do at all times, and there were very few disciplinary cases. Busy minds don't have time to think of mischief. (4) They learned. (5) The girls who helped me thought they, too, had learned—since teaching meant that they really had to know their work. They benefited by the added responsibility given them, and from the valuable teaching and social experience they gained. (6) I worked hard, too, but only a little longer than was the usual procedure. (7) I enjoyed the class.

Some questions that were left either partially or fully unanswered were: "Will the small-group method work in larger classes?" "Will it work with brighter groups?" "Will it work with higher grades?" "Is it worth the effort to make it work?" "Can we find student helpers who can command the respect necessary to do a good job?"

I believe that I have found some of the answers. Maybe I will find the others next term.

An honors program

by Norman Clark and Dick Lauder, Kirkland, Washington

A vigorous academic challenge extending beyond the usual high school curriculum is the purpose of an "honors program" at Lake Washington High School in Kirkland, Washington. The program, now on an experimental basis, takes students of exceptional academic promise through mathematics, logic, science, composition, history, literature, and humanistic courses that provide intellectual stimulation not common in the public schools. The student body and faculty have given the program whole-hearted approval and public reaction so far has been highly favorable.

The basic premise of the program is that the individual satisfaction of solid intellectual accomplishment is indispensable to the development of these students. The social benefits gained from intensified training of the gifted follow as a logical consequence. The faculty feels that if the public schools are to be truly democratic they must provide this type of intellectual satisfaction derived from intense training.

Sophomores in the honors course have a two-hour period in world history with related readings in world literature, coupled with the English composition normally taught in grade eleven. Their mathematics, a one-hour course, covers plane geometry and advance algebra. As juniors the students have a two-hour period in American history and literature, composition through grade thirteen, and a one-hour mathematics class in trigonometry and analytical geometry. The seniors take their two-hour period in physics and the calculus; they have a one-hour course in contemporary humanistic studies and intensified composition review.

James McKeehan, the mathematics teacher, finds that even tenth- and eleventh-graders are capable of working with such topics as number theory, number systems, the algebraic theory of sets, geometrical constructions, projective, axiomatic, and non-Euclidean geometries—all self-contained logical systems which require no mastery of tool subjects beyond the scope of the program.

Mr. McKeehan has learned that by introducing such topics and not striving for their mastery, he can stimulate the interest of the pupils by showing them the beauty, the power, and the scope of the ideas of modern mathematics. They are fascinated not only by t[illegible] of advanced study, but by the intellectual world revealed. Last year one student, stimulated by the course, developed his own project on the area of a polygon in terms of its vertices, entered the Westinghouse Talent Search, and won. The group with which this boy was associated studied functions and limits, limits and continuity, the calculus, and the calculus of variations.

The textbooks used include Courant and Robbins, *What Is Mathematics; An Elementary Approach;* Brixey and Andree, *An Introduction to College Mathematics;* E. T. Bell, *Men of Mathematics;* and Kline, *Mathematics and Western Culture.*

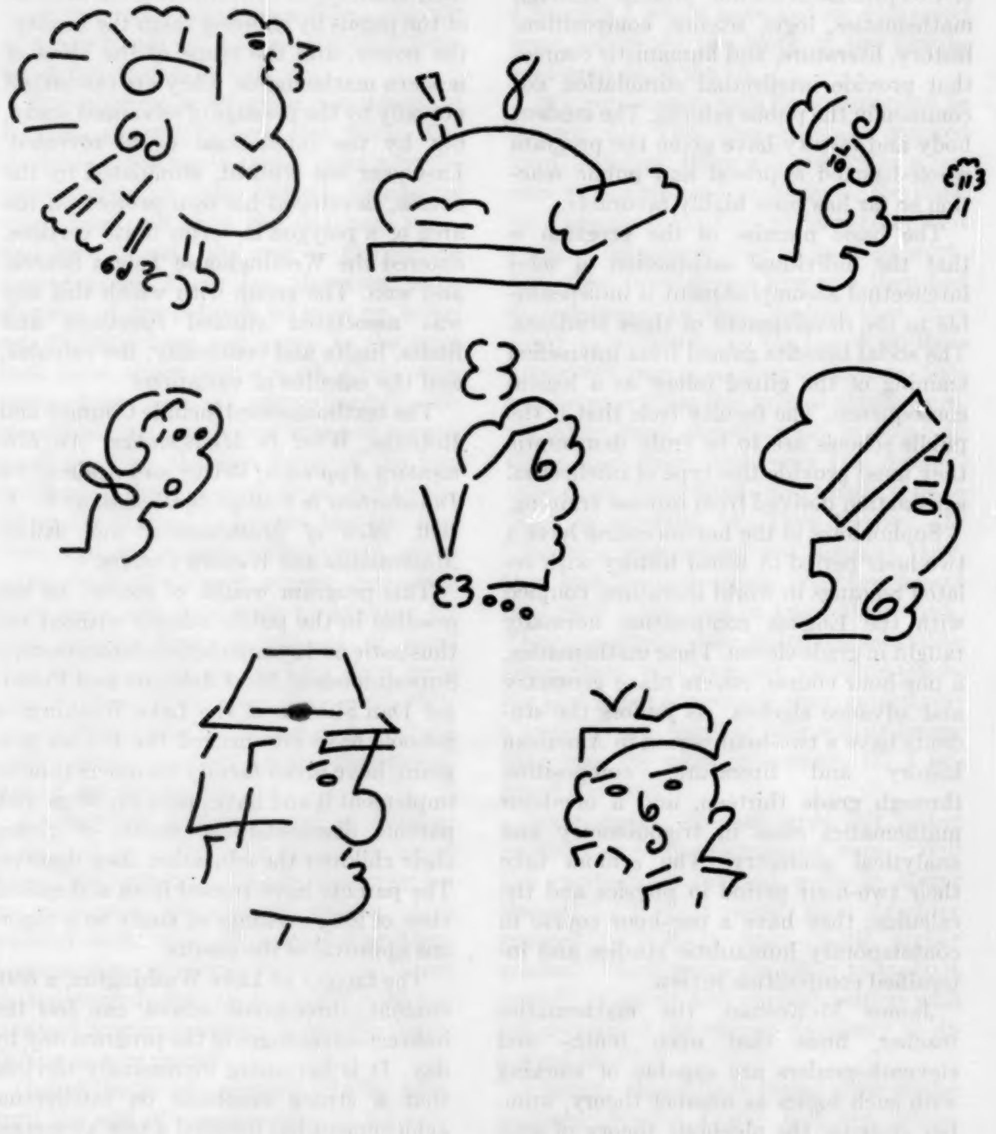
This program would, of course, be impossible in the public schools without enthusiastic and sympathetic administrators. Superintendent Mort Johnson and Principal Dan Sho[illegible]n of the Lake Washington Schools have encouraged the honors program, have given faculty members time to implement it and have spent evenings with parents discussing the merits of giving their children the education they deserve. The parents have turned from a skeptical view of long evenings of study to a vigorous approval of the results.

The faculty at Lake Washington, a 650-student, three-grade school, can feel the indirect advantages of the program day by day. It is becoming increasingly obvious that a strong emphasis on intellectual achievement has instilled a new awareness of humanistic values in the entire school.

Number cartoons

by George Janicki, Elm School, Elmwood Park, Chicago, Illinois

The eighth grade class at Elm School, Chicago, Illinois took great interest in making original number cartoons. The class had great fun finding the hidden numbers and their sum. Try your luck.



NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Fifteenth Summer Meeting

Indiana University, Bloomington, Indiana

August 21, 22, 23, 24, 1955

THEME

What, how, and why in mathematics

The Fifteenth Summer Meeting of The National Council of Teachers of Mathematics will begin informally on Saturday, August 20, 1955, with a day of activities in Indianapolis, Indiana. These activities will include a free luncheon for Council members (and their families) at the Allison Division of General Motors and a trip through their Powerama, a visit to the planetarium at Butler University, and a visit to the Indianapolis Speedway, home of the famous 500-mile race.

(Although Council members will be guests of General Motors for luncheon and Powerama trip, the hosts will need to know the number of individuals who will attend. *Make reservations in advance* with Mrs. Eleanor Guyer, 71 South Street, Southport, Indiana.)

On Sunday, August 21, the scene will shift 50 miles south and west to the campus of Indiana University at Bloomington. There the proceedings will assume a more "official" status, with registration starting at 3:00 P.M. and continuing throughout the remainder of the afternoon and into the evening.

The convention proper will convene at 8:45 A.M. on Monday, August 22, and run through the noon luncheon on Wednesday, August 24.

LOCALE

Southern Indiana represents one of the major vacation areas in the Central States. Sixteen miles east of Bloomington is Brown County State Park with swimming, horseback riding, fishing, and wonderful scenery; 20 miles west is McCormick's Creek State Park which is not quite so large but has facilities similar to those of Brown County. About 32 miles south is Spring Mill State Park, which combines beautiful scenery with many historical shrines of the developing West, such as a mill which still grinds meal on stone burrs by water power and several cottages which are reproductions of the pioneer type, with furniture of that period. Swimming and other recreational facilities are also available.

Bloomington itself is the center of the territory which produces the famous Indiana limestone—stone which graces innumerable public (and private) buildings throughout the nation, including the Pentagon in Washington and the Woolworth and Rockefeller Center buildings in New York.

ACCOMMODATIONS

For the members of the National Council who attend the Summer Meeting,

housing will be available in the New Smithwood. This is a strictly modern 600-room dormitory, just now in the process of being completed. There are both double and single rooms available and there is a half-bath for each of two rooms. The furniture, of course, is all new. There are lounges and recreation rooms, as well as a sun deck.

Accommodations will be on the American plan at $5.50 or $6.00 per day. Meals will be served in the Smithwood dining hall.

For families with children, special rates will be provided.

New Smithwood is located directly across the street from the School of Education building where all the meetings of the Council will be held. The School of Education building is air conditioned and you will be able to participate in Council programs in complete comfort. Surrounding New Smithwood there will be ample parking space for all individuals who wish to drive their cars; the city buses pass directly in front of the building every 20 minutes to transport you to downtown theaters, parks, and swimming pool. Provision will be made for National Council participants who wish to use the facilities of the University Library.

Program

The Program for Monday, August 22, and Wednesday, August 24, will follow the customary pattern of a general address followed by sectional meetings and discussion groups. Tuesday's program will approximate the *continuity* sessions featured in previous conventions: major morning addresses will be the subjects for panel discussions in the afternoon.

Featured *general* addresses include: "More Mathematics?" by John R. Mayor, University of Wisconsin, Madison, Wisconsin; "Modern Algebra Belongs in the High School," by Saunders MacLane, University of Chicago, Chicago, Illinois; and "Present and Future Industrial Data Processing Installations" by John T. Horner, Supervisor, Engineering Calculation Group, Allison Division, General Motors Corporation, Indianapolis, Indiana.

Featured activities include a picnic at Brown County State Park followed by a play at the Brown County Play House on Monday evening, a banquet on Tuesday evening, and a luncheon on Wednesday.

Information

Information concerning local arrangements may be secured by writing to the local chairman, Dr. Philip Peak, School of Education, Indiana University, Bloomington, Indiana.

Programs will be mailed only to members living in or near Indiana. Others may secure copies by writing to Myrl Ahrendt, Executive Secretary, 1201 Sixteenth Street, N.W., Washington 6, D. C.

"I was just going to say, when I was interrupted, that one of the many ways of classifying minds is under the heads of arithmetical and algebraical intellects. All economical and practical wisdom is an extension or variation of the following arithmetical formula: $2+2=4$. Every philosophical proposition has the more general character of the expression $a+b=c$. We are mere operatives, empirics, and egotists, until we learn to think in letters instead of figures."—*Oliver Wendell Holmes*, The Autocrat of the Breakfast Table, *page* 1.

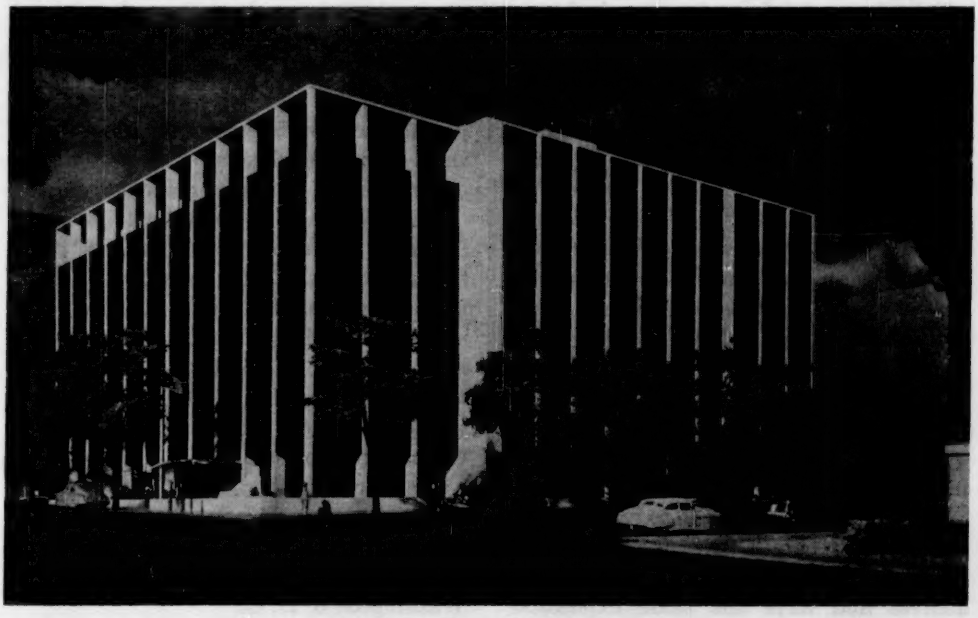
Edited by M. H. Ahrendt, Executive Secretary, NCTM, Washington, D. C.

New NEA Educational Center

by M. H. Ahrendt

The adjoining picture shows the architect's conception of the new NEA Educational Center. On famous Sixteenth Street, five blocks from the White House, near the offices of other national educational and scientific societies, the new center is being built on property now owned by the NEA. The new building will be one of the most significant milestones of professional educational progress of this generation. It will be, in a city filled with symbols and monuments, a lasting symbol of the importance of education in our national life.

The present headquarters of the NEA is adequate for only 200 employees. But the staff now numbers more than 500. To provide housing for the excess, six buildings outside the main offices have been acquired: a hotel, a warehouse, a commercial garage, and three residences. The work of the staff is hampered by cramped quarters. Creative work is difficult amid the clattering of typewriters and other office noises. Messengers are required for shuttling routine projects from building to building. Even so simple a service as mimeographing a sheet of paper requires a round trip of twelve blocks to the duplicating rooms temporarily sheltered in a warehouse. To provide the space needed for creative and efficient work, the new building is imperative.



The NEA Educational Center

The construction of the new building has special significance to The National Council of Teachers of Mathematics. On March 17 the headquarters office of the Council was moved to Room 701 in the new building. We consider ourselves fortunate indeed to be among the first occupants. The additional space, the more pleasant surroundings, and the summer air conditioning are lifting the spirits of the staff of the Washington Office.

Members of the Council can assist with the building program in a number of ways. Your most valuable contribution would be to buy a life membership in the NEA. All proceeds from life memberships are being reserved for the building program. Special contributions of any size are welcome. Memorial gifts are appropriate. Anything you can do to increase the membership of the NEA will help.

The new Center was originally scheduled to be completed in 1957. The building program involves three stages. Stage one is nearing completion. The work on stage two will then begin. Whether the work can be continued until stage three is completed will depend on the response of the teaching profession. To date, the response has been very encouraging.

NCTM membership to be given to Arithmetic Teacher subscribers

by M. H. Ahrendt

Beginning with the 1955–56 school year, subscribers to *The Arithmetic Teacher* will be given membership privileges in The National Council of Teachers of Mathematics and the frequency of publications will be increased from four to six issues per year.

The new plan calls for a uniform membership fee of $3.00. This fee will cover full membership privileges and benefits, including the rights to vote and hold office, and a subscription to either *The Mathematics Teacher* or *The Arithmetic Teacher*. Those who wish to receive both journals may have them for the special price of $5.00, provided both journals are sent to the same address and have the same expiration date.

The student membership fee, which is available to any student who has never taught, will remain at $1.50, with student members also having a choice of journals. Student members who wish to receive both journals may obtain them for $2.50, provided both journals are sent to the same address and have the same expiration date. (Student membership does not include the rights to vote and hold office.)

The change in fee will go into effect on *The Arithmetic Teacher* subscriptions or renewals for which mailing begins with the October 1955 issue. Persons who wish to receive both journals at the reduced joint rate should arrange to have their subscriptions run for the same period of time. In cases in which a person now receives both journals with different expiration dates, a short-term subscription will be accepted in order to make the dates coincide.

The rates to institutions (schools, libraries, departments, etc.) will be $5.00 for each journal. Common expiration dates are not necessary for institutions that subscribe to both journals. An additional mailing charge of 25¢ to Canada and 50¢ to foreign countries is required for either journal on both individual and institutional subscriptions.

For further information write M. H. Ahrendt, Executive Secretary, The National Council of Teachers of Mathematics, 1201 Sixteenth Street N.W., Washington 6, D. C.

Membership in The National Council of Teachers of Mathematics, February 1, 1955

	Individual	Institutional	Total	February 1, 1954
Albama	75	35	110	119
Arizona	25	14	39	43
Arkansas	76	20	96	111
California	351	221	572	562
Colorado	119	22	141	118
Connecticut	136	37	173	150
Delaware	36	6	42	46
District of Columbia	108	16	124	123
Florida	180	44	224	211
Georgia	97	34	131	123
Idaho	7	4	11	14
Illinois	706	102	808	778
Indiana	291	44	335	364
Iowa	151	48	199	210
Kansas	185	35	220	199
Kentucky	52	19	71	79
Louisiana	146	37	183	169
Maine	40	9	49	43
Maryland	168	26	194	185
Massachusetts	253	53	306	288
Michigan	297	96	393	379
Minnesota	176	70	246	255
Mississippi	59	23	82	86
Missouri	173	49	222	216
Montana	40	9	49	38
Nebraska	80	22	102	112
Nevada	11	6	17	9
New Hampshire	27	9	36	43
New Jersey	302	78	380	408
New Mexico	32	18	50	56
New York	622	199	821	782
North Carolina	131	45	176	158
North Dakota	20	11	31	28
Ohio	420	83	503	497
Oklahoma	122	46	168	152
Oregon	73	31	104	89
Pennsylvania	444	180	624	603
Rhode Island	38	9	47	47
South Carolina	42	37	79	91
South Dakota	22	8	30	26
Tennessee	107	41	148	161
Texas	332	138	470	451
Utah	25	15	40	32
Vermont	20	5	25	27
Virginia	208	49	257	237
Washington	124	36	160	151
West Virginia	70	10	80	105
Wisconsin	233	63	296	266
Wyoming	19	9	28	26
TOTALS	7,471	2,221	9,692	9,466
U. S. Possessions	25	14	39	36
Canada	85	63	148	139
Foreign	111	99	210	216
GRAND TOTALS	7,692	2,397	10,089	9,857

• POINTS AND VIEWPOINTS

A column of unofficial comment

National Council news

by Marie S. Wilcox, President, The National Council Teachers of Mathematics

The Board of Directors of the National Council is happy to announce that John R. Clark has been added to the staff of *The Arithmetic Teacher*. With the addition of a second associate editor, Dr. Ben Sueltz and his staff will publish a larger number of issues of *The Arithmetic Teacher* per year. Beginning with the October issue, subscriptions to this journal, as such, will not be available. A member will pay regular dues of $3.00 a year and receive whichever journal he designates, *The Arithmetic Teacher* or The Mathematics Teacher. Membership including both journals will be available at $5.00 per year. During the summer months the Executive Office of the Council will make the proper adjustments for members who wish to make some change in the journals they receive.

Another recent action of the Board of Directors sets the dates for the expiration of the terms of the editors of Council journals so that not more than one position will become vacant in one year. In order to provide for this schedule agreement was reached on the following dates. The present term of Henry Van Engen, editor of THE MATHEMATICS TEACHER, will expire on June 1, 1956; that of Ben Sueltz, editor of *The Arithmetic Teacher*, will expire on June 1, 1957; and that of Harold Larsen, editor of *The Mathematics Student Journal*, will expire on June 1, 1958. According to the constitution, each editor serves a term of three years and may be reappointed, but may not serve for more than two consecutive terms.

The constitution provides that "the nine directors shall be elected by members of the Council according to a geographic plan determined by the Board." Some years ago the Board made a regulation that not more than one person from a state should serve on the Board at the same time. This has usually given a good geographic distribution on the Board. For instance, the Board as constituted during the year preceding the Boston meeting included two members from the Northeast, two from the Southeast, two from the Middle West, one from the Southwest, and two from the West coast. However, at the St. Louis meeting of the Board, further action was taken to assure geographic distribution. The Board action now provides that "Nominations for directors shall be made so that there shall not be more than one director elected from each state; also, there shall be one director, and not more than two, elected from each area of the Affiliated Groups." There had been five areas from the Affiliated Groups and these have now been increased to six.

The Board also took action to assist the Nominating Committee. This committee will hold at least two meetings, and names of persons to be considered for nomination will be secured in a variety of ways so that each area and each group will have an opportunity to present names of prospective candidates.

Early this year Affiliated Groups were invited to send names to the nominating committee. At the Delegate Assembly a second invitation was extended to these groups to present names, and a box was

provided at the annual meeting in which any member might place suggestions for possible nominees. Also, in this issue, of The Mathematics Teacher, the chairman of the nominating committee is requesting names from the membership at large.

The National Council continues to cooperate with other organizations. Beginning last year, officers of NEA departments were invited to a meeting in Washington to discuss mutual interests and problems. This year the meeting was held on May 9 and 10. The President and Executive Secretary of the National Council attended the meeting.

During the past school year, we have been invited to send representatives to meetings of various other organizations. Francis G. Lankford, Jr., of the University of Virginia represented us at the meeting of the Council of Rural Education. Frank B. Allen of LaGrange, Illinois was our representative at the Council on Cooperation in Teachers Education.

Henry Syer of Boston University is our new representative on the Cooperative Committee of the American Association for the Advancement of Science.

1956 Election

The new Committee on Nominations and Elections is already making plans for the next election in 1956. Officers to be elected in the 1956 election will be as follows: president, vice-president for high school, vice-president for elementary, and three directors.

The committee is anxious to receive your suggestion for candidates. If you know of able individuals who should be considered for office, please submit their names now with full information about their qualifications to the new chairman of the Committee on Nominations and Elections, Maurice L. Hartung, University of Chicago, Chicago 37, Illinois.

Annual meeting

The summer meeting of the National Council of Teachers of Mathematics, in co-operation with the National Education Association, will be held in Wieboldt Hall Monday, July 4, 1955, at the Chicago Avenue Campus of Northwestern University in Chicago, Illinois.

The morning program will concern the use of calculating devises from the abacus to the electronic computer. Registration will start at 9:00 A.M.

Luncheon will be available in the cafeteria in Abbott Hall.

There is no formal program in the afternoon but practice sessions are scheduled at 1:30 P.M. and 2:45 P.M. when those interested may learn to use and practice using the abacus, calculating machines, or the slide rule. Registration is necessary for these afternoon laboratory sessions. Those persons who wish to participate should contact E. H. C. Hildebrandt, 221 Lunt Building, Northwestern University, Evanston, Illinois.

It is not fair

by Henry Van Engen, Iowa State Teachers College, Cedar Falls, Iowa

Comments on the present "bad" state of high school mathematics often make references to the reduction in mathematical requirements in the high school. This reduction is mentioned in such a "tone" that the inference is drawn—if not stated explicitly—that the reduction in requirements is in large part responsible for the present state of affairs.

A second comment one hears, at times, about mathematical situations in the high school is to the effect that guidance functions very badly. It is implied that many good students go through high school without having been told that they should study mathematics. This may or may not be true. Statistics are never offered. Certainly, like all things humans do, guidance

techniques and practices could be improved.

In the light of such comments it is interesting to get some statistics—actual numbers, not opinions. To this end we quote from the *Spotlight on Organization and Supervision in Large High Schools*, a United States Department of Health, Education and Welfare publication.

It is not fair to compare the number of pupils studying Latin (or any other HS subject) with the number studying it 50 years ago, because the total HS enrollment has zoomed from 202,963 (1890) to 7,688,919 (1952, incl. Jr. HS). Or look at it this way: Latin enrollment in 1890 was 70,411 students whereas 1949 registration reached 422,304; percentagewise, however, the corresponding figures are 34.7 and 7.8 percent respectively.

It *is* fair to compare HS subject registrations by percentage of pupils enrolled in specific subjects. But comparison is possible only for a few subjects, as many now offered did not exist 50 years ago. The following table tells the story.

PERCENTAGE OF TOTAL PUPILS, GRADES 9 TO 12, IN PUBLIC HIGH SCHOOLS, ENROLLED IN SUBJECT AREAS, 1948–49 AND 1889–90

Subject	1948–49	1889–90
English	94.9%	not reported
Physical Education	69.4%	0*
Social Studies	62.3%	27.3%
Business Education	58.1%	0
Mathematics	54.7%	66.7%
Science	53.8%	32.9%
Music	30.1%	0
Industrial Arts	26.6%	0
Home Economics	24.2%	0
Foreign Languages	14.0%	16.3%
Art	9.0%	0
Latin	7.8%	34.7%
Agriculture	6.7%	0

* Indicates no enrollment.

Sources: Offering and Enrollments in High School Subjects, 1948–49, Table 7, p. 107–8. Biennial Survey of Education in the US, 1948–50. Government Printing Office, Washington 25, D. C.; price 30 cents. Statistics of Public Secondary Day Schools, 1951–52, Biennial Survey of Education in the US, 1950–52. Table 6, p. 52. Government Printing Office; price 35¢.

Note, in particular, the change in enrollment from 1890 to 1952. Also note the change in percentage of high school pupils taking mathematics. While the enrollment increased more than 2,500%, the per cent of pupils taking mathematics decreased from 66.7% to 54.7%. This, in spite of the fact that a decrease in general ability accompanied the increase in enrollment.

M. L. Hartung, in a discussion of current criticisms of education pointed out:

"In relative enrollment figures usually used, the base is the total number of pupils in the last four years of public secondary schools. It is interesting to study the situation by using as a base the total population ranging from fourteen to seventeen years in age. The resulting per cents give an indication of the extent to which the public schools have made the study of certain subjects available to the *total youth population of appropriate age*. In 1900, only 4.7% of that population were enrolled in algebra. In 1949, the figure had increased to 17% . In 1900, 1.6% of the country's youth were taking physics. In 1949, the figure was 3.4%. Chemistry changed from less than 1% to 4.8% during the same period."[1]

The table of figures from the *Spotlight* yields other interesting facts. Note the foreign languages. In 1890, 16.3% of the high school students were taking a foreign language. In 1949, 14.0% of the students were taking foreign language. Has the high school neglected foreign languages? Maybe, but if they have, then foreign languages must have been neglected in 1890 also. However, note the change in the figures for Latin.

Little by little comparisons are being made and the schools in the mid-century stand up pretty well.[2] One can confidently say that the data does not warrant the immense amount of criticism the schools receive. This does not mean, however, that the schools—colleges and high schools—do not have a lot of work to do in the next few years. Changes of a radical kind must come, and not the least of these is the introduction of a more modern type of mathematics in the high schools and the colleges.

[1] M. L. Hartung, "Modern Methods and Current Criticisms of Mathematical Education," *School Science and Mathematics*, Vol. XLV (February 1955), p. 86.

[2] C. V. Newsom, "Some Educational Problems of Significance to Engineering Colleges," THE MATHEMATICS TEACHER, Vol. XLVIII (April 1955), p. 194.

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D.C.

NCTM convention dates

July 4, 1955
JOINT MEETING WITH NEA
Chicago, Illinois
E. H. C. Hildebrandt, local chairman, Northwestern University, Evanston, Illinois

August 21–24, 1955
SUMMER MEETING
Indiana University, Bloomington, Indiana
Philip Peak, local chairman, Indiana University, Bloomington, Indiana

December 27–30, 1955
CHRISTMAS MEETING
Sheraton-Park Hotel, Washington, D. C.
Veryl Schult, local chairman, Wilson Teachers College, Washington 9, D. C.

April 11–14, 1956
ANNUAL MEETING
Schroeder Hotel, Milwaukee, Wisconsin
Margaret Joseph, 1504 N. Prospect Avenue, Milwaukee 2, Wisconsin

Other professional dates

University of Oklahoma Institute for Teachers of Mathematics
June 6–17, 1955
University of Oklahoma, Norman, Oklahoma
J. O. Hassler, University of Oklahoma, Norman, Oklahoma

Summer Institute for Teachers of Secondary School Mathematics
June 13–July 23, 1955
Oklahoma A. and M. College, Stillwater, Oklahoma
James H. Zant, Oklahoma A. and M. College, Stillwater, Oklahoma

Ninth Annual Conference in Elementary Education—Arithmetic
June 14–17, 1955
Ohio University, Athens, Ohio
Holbert H. Hendrix, Ohio University, University Elementary School, Athens, Ohio

Sixth Annual Mathematics Institute
June 19–24, 1955
Louisiana State University, Baton Rouge, Louisiana
Houston T. Karnes, Louisiana State University, Baton Rouge 3, Louisiana

Workshop on Elementary Mathematics
June 20–July 1, 1955
Iowa State Teachers College, Cedar Falls, Iowa
I. H. Brune, Iowa State Teachers College, Cedar Falls, Iowa

Conference on Secondary Mathematics
June 20–July 1, 1955
Iowa State Teachers College, Cedar Falls, Iowa
H. Van Engen, Iowa State Teachers College, Cedar Falls, Iowa

Eighth Annual Workshop for Teachers of Mathematics
June 20–July 2, 1955
Indiana University, Bloomington, Indiana
Philip Peak, School of Education, Indiana University, Bloomington, Indiana

Summer Conference for Mathematics Teachers
June 27–July 22, 1955
University of Wisconsin, Madison, Wisconsin
C. C. MacDuffee, Director, 302 North Hall, University of Wisconsin, Madison 6, Wisconsin

California Conference for Teachers of Mathematics
July 5–15, 1955
University of California, Los Angeles, California
Clifford Bell, University of California, Los Angeles 24, California

Institute for Mathematics Teachers
August 1–12, 1955
University of Virginia, Charlottesville, Virginia
F. G. Lankford, Jr., University of Virginia, Charlottesville, Virginia

Third Annual Mathematics Institute of the Florida Council of Teachers of Mathematics
August 18–20, 1955
University of Florida, Gainesville, Florida
Kenneth P. Kidd, University of Florida, Gainesville, Florida